A THEOREM FOR FOURIER COEFFICIENTS OF A FUNCTION OF CLASS L^p

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Abstract: This paper deals with the Fourier coefficients of a function of class L^p . We give a necessary and sufficient condition for a function to be of class L^p for p greater than one.

Key Words and Phrases: Fourier coefficients, L^p class, and monotonically decreasing.
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1. INTRODUCTION.

A function $f(x)$ is said to belong to the class $L(p, \alpha)$ if $\int_0^\pi |f(x)|^p (\sin x)^{\alpha p} dx < \infty$ [1].

If $f(x) \in L(p, \alpha)$, then we define $\|f\|_{p, \alpha} = \left\{ \int_0^\pi |f(x)|^p (\sin x)^{\alpha p} dx \right\}^{\frac{1}{p}}$.

Hardy [2] gave the following theorem concerning the Fourier coefficients of a function belonging to L^p class.

THEOREM 1.1: Let $a_1, a_2, \dots$ be Fourier cosine coefficients of a function of class L^p , $p \geq 1$, and $s_n = \sum_{k=1}^n a_k$.

Then $\frac{s_1}{1}, \frac{s_2}{2}, \frac{s_3}{3}, \dots$ are also Fourier coefficients of a function of class L^p .

The converse of Theorem 1.1 is not necessarily true. But Siddiqui [3] proved the following theorem.

THEOREM 1.2: Let $f(x) \approx \sum_{n=1}^\infty a_n \cos nx$ with $a_n \not\equiv 0$. Then a necessary and sufficient condition that

$\sum_{n=1}^\infty a_n \cos nx$ be the Fourier series of $f(x) \in L^p$ is that $\sum_{n=1}^\infty A_n \cos nx$ be the Fourier series of a function

belonging to L^p class, where $p > 1$ and $A_n = \frac{1}{n} \sum_{k=1}^n a_k$.

2. MAIN RESULT. The object of this paper is to weaken the hypothesis that $a_n \not\equiv 0$ of Theorem 1.2 to a condition that $n^{-\beta} a_n$ should be monotonic for some non-negative integer β and also for weighted L^p spaces. In fact we have the following theorem.

THEOREM 2.1: Let $\{a_n\}$ be a positive null sequence such that $n^{-\beta} a_n$ is monotonically decreasing for some non-negative integer β . Suppose $f(x) \approx \sum_{n=1}^\infty a_n \cos nx$. Then a necessary and sufficient condition

that the series $\sum_{n=1}^{\infty} a_n \cos nx$ be the Fourier series of $f(x) \in L(p, \alpha)$ is that $\sum_{n=1}^{\infty} A_n \cos nx$ be the Fourier

series of a function belonging to $L(p, \alpha)$ class, where $1 \leq p < \infty$, $-1 < \alpha < p-1$ and $A_n = \frac{1}{n} \sum_{k=1}^n a_k$.

We shall require the following Lemmas for the proof of our theorem.

LEMMA 2.2 [1]: Let $f(x) \approx \sum_{n=1}^{\infty} a_n \cos nx$ where the a_n are positive and tend to zero and $n^{-\beta} a_n$ is

monotonically decreasing for some non-negative integer β . Then a necessary and sufficient condition

that $f(x) \in L(p, \alpha)$ is that $\sum_{n=1}^{\infty} n^{p-\alpha p-2} a_n^p < \infty$ where $1 \leq p < \infty$ and $-1 < \alpha < p-1$.

LEMMA 2.3: If $n^{-\beta} a_n$ for some non-negative integer β is monotonically decreasing, then

$$A_n = \frac{\frac{1}{n} \sum_{k=1}^n a_k}{n^{\frac{1}{\beta}}}$$

is also monotonically decreasing.

Proof: Let $\beta=0$, then we have to show that

$$A_n = \frac{1}{n} \sum_{k=1}^n a_k \geq A_{n+1} = \frac{1}{n+1} \sum_{k=1}^{n+1} a_k,$$

$$\text{or} \quad (n+1) \sum_{k=1}^n a_k \geq n \sum_{k=1}^{n+1} a_k$$

$$\text{or} \quad n \sum_{k=1}^n a_k + \sum_{k=1}^n a_k \geq n \sum_{k=1}^n a_k + n a_{n+1},$$

$$\text{or} \quad n a_{n+1} \leq \sum_{k=1}^n a_k.$$

Since

$$a_{n+1} \leq a_1,$$

$$a_{n+1} \leq a_2,$$

$$\vdots$$

$$a_{n+1} \leq a_n,$$

it follows that

$$n a_{n+1} \leq a_1 + a_2 + \dots + a_n,$$

$$\text{or} \quad n a_{n+1} \leq \sum_{k=1}^n a_k.$$

Thus $A_n \geq A_{n+1}$.

Now let $\beta \geq 1$. Let $C_n = n^{-\beta} a_n$, then

$$\begin{aligned} A_n &= \frac{1}{n} \sum_{k=1}^n a_k = n^{-(1+\beta)} \sum_{k=1}^n a_k \\ &= n^{-(1+\beta)} \sum_{k=1}^n k^\beta \frac{a_k}{k^\beta} \\ &= n^{-(1+\beta)} \sum_{k=1}^n k^\beta C_k, \end{aligned}$$

and

$$\begin{aligned} A_{n+1} &= (n+1)^{-(1+\beta)} \sum_{k=1}^{n+1} k^\beta C_k \\ &= (n+1)^{-(1+\beta)} \left\{ \sum_{k=1}^n k^\beta C_k + (n+1)^\beta C_{n+1} \right\} \\ &= (n+1)^{-(1+\beta)} \sum_{k=1}^n k^\beta C_k + (n+1)^{-1} C_{n+1}. \end{aligned}$$

Now

$$\begin{aligned} A_n - A_{n+1} &= n^{-(1+\beta)} \sum_{k=1}^n k^\beta C_k - (n+1)^{-(1+\beta)} \sum_{k=1}^n k^\beta C_k - (n+1)^{-1} C_{n+1} \\ &= \{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \sum_{k=1}^n k^\beta C_k - (n+1)^{-1} C_{n+1} \end{aligned}$$

therefore

$$\begin{aligned} (n+1)(A_n - A_{n+1}) &= (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \times \sum_{k=1}^n k^\beta C_k - C_{n+1} \\ &\geq (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} C_n \sum_{k=1}^n k^\beta - C_{n+1} \\ &\geq (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} C_{n+1} \sum_{k=1}^n k^\beta - C_{n+1} \\ &= C_{n+1}[(n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \sum_{k=1}^n k^\beta - 1] \\ &= C_{n+1}\{(n+1)\theta_n - 1\}, \text{ where} \\ (n+1)\theta_n &= (n+1)\{n^{-(1+\beta)} - (n+1)^{-(1+\beta)}\} \sum_{k=1}^n k^\beta \\ &= \{(n+1)n^{-(1+\beta)} - (n+1)(n+1)^{-(1+\beta)}\} \sum_{k=1}^n k^\beta \\ &= \{(n+1)n^{-(1+\beta)} - (n+1)^{-\beta}\} \sum_{k=1}^n k^\beta \end{aligned}$$

$$\begin{aligned}
&= \{(n+1)n^{-1}n^{-\beta} - n^{-\beta}(1+\frac{1}{n})^{-\beta}\} \sum_{k=1}^n k^{\beta} \\
&= n^{-\beta} \{n(1+\frac{1}{n})n^{-1} - (1+\frac{1}{n})^{-\beta}\} \sum_{k=1}^n k^{\beta} \\
&= n^{-\beta} \{(1+\frac{1}{n}) - (1+\frac{1}{n})^{-\beta}\} \sum_{k=1}^n k^{\beta} \\
&= n^{-\beta} \{(1+\frac{1}{n}) - (1+\frac{1}{n})^{-\beta} + \frac{\beta(\beta-1)}{2} \cdot \frac{1}{n^2} \dots\} \times \sum_{k=1}^n k^{\beta} \\
&= n^{-\beta} \{\frac{1}{n} + \frac{\beta}{n} - \frac{\beta(\beta-1)}{2} \cdot \frac{1}{n^2} + \dots\} \sum_{k=1}^n k^{\beta}
\end{aligned}$$

Now by the following formula in [4],

$$\sum_{k=1}^n k^{\beta} = \frac{n^{\beta+1}}{\beta+1} + \frac{n^{\beta}}{2} + \frac{\beta \cdot n^{\beta-1}}{12} - \frac{\beta(\beta-1)(\beta-2)}{720} n^{\beta-3} + \dots$$

we have

$$\begin{aligned}
(n+1)\theta_n &= 1 + \frac{1}{2}\{(\beta+1) - \frac{\beta(\beta+1)}{\beta}\}n^{-1} + O(n^{-2}) \\
&= 1 + \frac{(3\beta+1)}{(\beta+1)}n^{-1} + O(n^{-2})
\end{aligned}$$

>1 for large n.

LEMMA 2.4: Let $\{a_n\}$ be a positive sequence which tends to zero. Let $\{n^{-\beta} a_n\}$ be monotonically decreasing for some non-negative integer β . Then the convergence of $\sum_{n=1}^{\infty} n^{p-\alpha p-2} A_n^p$ implies the convergence of the series

$$\sum_{n=1}^{\infty} n^{p-\frac{1}{\alpha p}-2} a_n^p,$$

where

$$A_n = \frac{1}{n} \sum_{k=1}^n a_k.$$

Proof: Since $\{n^{-\beta} a_n\}$ is a monotonically decreasing sequence, then it follows that

$$\begin{aligned}
A_n &= \frac{1}{n} \sum_{k=1}^n a_k = \frac{1}{n} \sum_{k=1}^n k^{-\beta} a_k k^{\beta} \\
&\geq \frac{1}{n} n^{-\beta} a_n \sum_{k=1}^n k^{\beta} = K a_n, \text{ for some } K
\end{aligned}$$

so that

$$\sum_{n=1}^{\infty} n^{p-\alpha p-2} a_n^p \leq K \sum_{k=1}^{\infty} n^{p-\alpha p-2} A_n^p < \infty$$

and hence the result follows.

Proof of the Theorem 2.1: The necessary part follows from Theorem B as a particular case.

Sufficiency. Since $\{A_n\}$ is a positive null sequence and due to Lemma 2.3, $\{n^{-\beta} A_n\}$ is monotonically decreasing for some non-negative integer β , it follows from Lemma 2.2 that if $\sum_{n=1}^{\infty} A_n \cos nx$ is the Fourier series of a function $F(x) \in L(p, \alpha)$, then $\sum_{n=1}^{\infty} n^{p-\alpha p-2} A_n^p < \infty$.

Applying Lemma 2.4, we have $\sum_{n=1}^{\infty} n^{p-\alpha p-2} a_n^p < \infty$.

Hence by Lemma 2.1, $f(x) \in L(p, \alpha)$, and consequently $\sum_{n=1}^{\infty} a_n \cos nx$ is the Fourier series of $f(x)$.

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