UNIFORM TOEPLITZ MATRICES

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ABSTRACT. We characterize all infinite matrices of bounded linear operators on a Banach space which preserve the limits of uniformly convergent sequences defined on an infinite set. Also, we give a Tauberian theorem for uniform summability by the Kuttner-Maddox matrix.

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1. INTRODUCTION.

By T we denote any infinite set of objects and we consider functions $f_k : T \rightarrow X$ for $k = 1, 2, \dots$, where $(X, ||\cdot||)$ is a Banach space.

The notation $f_k \Rightarrow f$ will be used to signify that $f_k \rightarrow f$ as $k \rightarrow \infty$, uniformly on T , which is to say that there exists $f : T \rightarrow X$ such that for all $\epsilon > 0$ there exists $k_0 = k_0(\epsilon) > 0$ with

$$||f_k(t) - f(t)|| < \epsilon, \text{ for all } k > k_0 \text{ and all } t \in T.$$

Now suppose that for $n, k = 1, 2, \dots$ each $A_{nk} \in B(X)$, i.e. each A_{nk} is a bounded linear operator on X . Then we shall say that $A = (A_{nk})$ is a uniform Toeplitz matrix of operators if and only if:

$$\sum_{k=1}^{\infty} A_{nk} f_k(t) \text{ converges in the norm of } X$$

for each $n \in N = \{1, 2, 3, 4, \dots\}$ and each $t \in T$ and

$$\sum_{k=1}^{\infty} A_{nk} f_k \Rightarrow f$$

whenever $f_k \Rightarrow f$.

Following Robinson [1] and Lorentz and Macphail [2], if (B_k) is a sequence in $B(X)$ we denote the group norm of (B_k) by

$$|| (B_k) || = \sup || \sum_{k=1}^p B_k x_k ||$$

where the supremum is over all $p \in \mathbb{N}$ and all x_k in the closed unit sphere of X .

By C we shall denote the $(C,1)$ matrix of arithmetic means, given by

$$\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & . & . & . \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & . & . & . \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{array}$$

By D we denote the Kuttner-Maddox matrix, used extensively in the theory of strong summability [3, 4, 5]:

$$\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \end{array}$$

In work on strong summability it is often advantageous to use the fact that, for non-negative (p_k) the summability methods C and D are equivalent, in the sense that $p_k \rightarrow 0(C)$ if and only if $p_k \rightarrow 0(D)$.

In connection with Tauberian theorems we now introduce the idea of uniform strong slow oscillation.

Let $s_k : T \rightarrow X$ for each $k \in \mathbb{N}$. Then we say that (s_k) has uniform strong slow oscillation if and only if $s_n - s_k \Rightarrow 0$ whenever $k \rightarrow \infty$ and $n > k$ with $n/k = O(1)$.

In what follows we shall regard s_k as the k -th partial sum of a given series of functions $\sum a_k = a_1 + a_2 + \dots$, each $a_k : T \rightarrow X$.

2. UNIFORM TOEPLITZ MATRICES.

The following theorem characterizes the uniform Toeplitz matrices of operators which were defined in Section 1.

THEOREM 1. $A = (A_{nk})$ is a uniform Toeplitz matrix if and only if

$$\sup_n || (A_{n1}, A_{n2}, \dots) || < \infty, \quad (2.1)$$

$$A \text{ is column-finite,} \quad (2.2)$$

$$\text{for each } n \in \mathbb{N}, A_n := \sum_{k=1}^{\infty} A_{nk} \text{ converges,} \quad (2.3)$$

$$A_n = I, \text{ ultimately in } n. \quad (2.4)$$

PROOF. We remark that in (2.3) the convergence is in the strong operator topology, and in (2.4), I is the identity operator on X .

For the sufficiency, let H denote the value of the supremum in (2.1), let

$n \in \mathbb{N}$ and $t \in T$. Then, for any $\varepsilon > 0$ there exists k_0 such that $||f_k(t) - f(t)|| < \varepsilon$ for all $k > k_0$.

Now for each $p \in \mathbb{N}$,

$$\sum_{k=1}^p A_{nk} f_k(t) = \sum_{k=1}^p A_{nk} (f_k(t) - f(t)) + \sum_{k=1}^p A_{nk} f(t),$$

where we assume that $f_k \Rightarrow f$. By (2.3), as $p \rightarrow \infty$, we have

$$\sum_{k=1}^p A_{nk} f(t) \rightarrow A_n f(t).$$

Also, if $s \geq r > k_0$,

$$||\sum_{k=r}^s A_{nk} (f_k(t) - f(t))|| \leq H\varepsilon,$$

whence $\sum_{k=1}^{\infty} A_{nk} f_k(t)$ converges.

By (2.4) there exists $m \in \mathbb{N}$ such that $A_n = I$ for all $n > m$, and by (2.2) there exists $n_0(\varepsilon) \in \mathbb{N}$ such that $A_{nk} = 0$ for $1 \leq k \leq k_0$ and for $n > n_0(\varepsilon)$.

Taking $n > m + n_0$ we have

$$\sum_{k=1}^{\infty} A_{nk} f_k(t) = f(t) + \sum_{k=1+k_0}^{\infty} A_{nk} (f_k(t) - f(t)).$$

Since

$$||\sum_{k=1+k_0}^{\infty} A_{nk} (f_k(t) - f(t))|| \leq \varepsilon ||(A_{n1}, A_{n2}, \dots)||,$$

it follows by (2.1) that $\sum A_{nk} f_k \Rightarrow f$, which proves the sufficiency.

Now consider the necessity. Take any convergent sequence (x_k) in X , with $x_k \rightarrow x$. Define $f_k(t) = x_k$ for all $k \in \mathbb{N}$ and all $t \in T$, and define $f(t) = x$ for all $t \in T$. Then $f_k \Rightarrow f$ and so $\sum A_{nk} x_k$ converges for each n and tends to x , whence the usual Toeplitz theorem for operators, see Robinson [1] or Maddox [6], yields (2.1) and (2.3) of our present theorem.

Next, suppose that (2.4) is false. Then there exist natural numbers $n(1) < n(2) < \dots$ with $A_{n(i)} \neq I$ for all $i \in \mathbb{N}$. Hence there exist $x_i \in X$ with

$$||A_{n(i)} x_i - x_i|| > 0 \quad (2.5)$$

for all $i \in \mathbb{N}$. Let us write $y(i)$ for the expression inside the norm bars in (2.5). Since T is an infinite set we may choose any countably infinite subset $\{t_1, t_2, t_3, \dots\}$ of T . Then we define $f: T \rightarrow X$ by

$$f(t_i) = x_i / ||y(i)|| \quad (2.6)$$

for all $i \in N$, and $f(t) = 0$ otherwise. If we define $f_k = f$ for all $k \in N$ then we certainly have $f_k \Rightarrow f$. But A is not a uniform Toeplitz matrix, since for $n = n(i)$ we have by (2.6),

$$\left| \left| \sum_{k=1}^{\infty} A_{nk} f(t_i) - f(t_i) \right| \right| = \left| \left| A_{ni} x_i - x_i \right| \right| / \left| \left| y(i) \right| \right| = 1.$$

Hence, if A is a uniform Toeplitz matrix then (2.4) must hold, and a similar argument shows that (2.2) is necessary, which completes the proof of the theorem.

Since C , the $(C,1)$ matrix, is not column-finite we immediately obtain:

COROLLARY 2. C is a Toeplitz matrix but not a uniform Toeplitz matrix.

However, since the elements of the Kuttner-Maddox matrix D are non-negative and its row sums all equal 1 it is clear that the conditions of Theorem 1 hold, whence D is a uniform Toeplitz matrix. Thus, whenever $f_k \Rightarrow f$ it follows that

$$2^{-r} \sum_r f_k \Rightarrow f, \quad (2.7)$$

where the sum in (2.7) is over $2^r \leq k < 2^{r+1}$ for $r = 0, 1, 2, \dots$. We also express (2.7) by writing $f_k \Rightarrow f(D)$.

The relation between C and D for uniform summability is given by:

THEOREM 3. $f_k \Rightarrow f(C)$ implies $f_k \Rightarrow f(D)$, but not conversely in general.

PROOF. Write

$$c(n) = n^{-1} \sum_{k=1}^n f_k(t) \quad \text{and} \quad d(r) = 2^{-r} \sum_r f_k(t).$$

Then we find that

$$d(r) = (2 - 2^{-r})c(2^{r+1} - 1) - (1 - 2^{-r})c(2^r - 1), \quad (2.8)$$

and it is clear that the right-hand side of (2.8) defines a uniform Toeplitz transformation between the c and d sequences.

For the last part of the theorem we may define real-valued functions on T by $f_k(t) = 2^r$ when $k = 2^r$ and $f_k(t) = -2^r$ when $k = 1 + 2^r$, and $f_k(t) = 0$ otherwise. Then $f_k \Rightarrow 0(D)$. Now suppose, if possible, that $f_k \Rightarrow f(C)$, which implies $f_k \Rightarrow f(D)$. Hence $f = 0$. But

$$c(2^r) - (1 - 2^{-r})c(2^r - 1) = 1,$$

contrary to the fact that $c(n) \rightarrow 0$.

3. A UNIFORM TAUBERIAN THEOREM.

By the remark following Corollary 2 we know that $f_k \Rightarrow f$ implies $f_k \Rightarrow f(D)$, but the example of Theorem 3 shows that the converse is generally false. The next result shows that uniform strong slow oscillation is a Tauberian condition for uniform D summability.

THEOREM 4. If (s_k) has uniform strong slow oscillation and $s_k \Rightarrow f(D)$ then $s_k \Rightarrow f$.

PROOF. Without loss of generality we may suppose that $f = 0$.

Take $n \in \mathbb{N}$ and determine r such that $2^r \leq n < 2^{r+1}$. If $\epsilon > 0$ there exists r_0 such that if $2^r \leq k < 2^{r+1}$ then

$$||s_k(t) - s_n(t)|| < \epsilon$$

whenever $r > r_0$ and $t \in T$. Since

$$2^{-r} \Sigma_r s_k(t) = s_n(t) + 2^{-r} \Sigma_r (s_k(t) - s_n(t))$$

we see that $s_n \Rightarrow 0$.

Our final result shows that the natural conditions $ka_k \Rightarrow 0$ or $ka_k \Rightarrow O(C,1)$ are both Tauberian conditions for uniform D summability, but that the restriction $ka_k \Rightarrow 0$ cannot be relaxed to the uniform boundedness of (ka_k) .

THEOREM 5. (i) If $ka_k \Rightarrow 0$ or $ka_k \Rightarrow O(C,1)$ and $s_k \Rightarrow f(D)$ then $s_k \Rightarrow f$.

(ii) There exists a divergent series Σa_k with (ka_k) uniformly bounded and $s_k \Rightarrow O(D)$.

PROOF. (i) First note that $ka_k \Rightarrow 0$ does not generally imply $ka_k \Rightarrow O(C,1)$ because $(C,1)$ is not a uniform Toeplitz matrix by Corollary 2. We shall show that $ka_k \Rightarrow O(C,1)$ is a Tauberian condition for D , the proof for $ka_k \Rightarrow 0$ being similar. In fact we shall show that $ka_k \Rightarrow O(C,1)$ implies that (s_k) has uniform strong slow oscillation.

Let us write $a_k = a_k(t)$, $s_n = s_n(t)$ and

$$A_n = n^{-1} \sum_{k=1}^n ka_k,$$

with the assumption that $A_n \Rightarrow 0$. Then for $n > k \geq 1$, by partial summation,

$$s_n - s_k = A_n - \frac{k}{n} A_k + \sum_{v=k+1}^{n-1} (vA_v - kA_k)/v(v+1),$$

whence

$$||s_n - s_k|| \leq \max\{||A_v|| : k \leq v \leq n\} (1 + \frac{k}{n} + 2 \sum_{v=k+1}^n \frac{1}{v+1}).$$

If $n/k = O(1)$ then

$$1 + \frac{k}{n} + 2 \sum_{v=k+1}^n \frac{1}{v+1} < 2 + 2 \frac{n}{k} = O(1),$$

and so $s_n - s_k \Rightarrow 0$, as required.

(ii) Define a numerical sequence (s_k) by $s_k = 0$ when $1 \leq k < 4$, and for $n \geq 2$

define $s_k = 0$ when $k = 2^n$ and when $k = 2^n + 2 \times 2^{n-2}$; $s_k = 1$ when $k = 2^n + 2^{n-2}$ and $s_k = -1$ when $k = 2^n + 3 \times 2^{n-2}$. Otherwise define s_k linearly, so that the graph of (s_k) is a triangular-shaped wave. Then (s_k) diverges and it is clear that $\sum_r s_k = 0$ for all $r \geq 0$. Also, it is easy to check that $k|a_k| \leq 8$ for all $k \geq 1$, whence our result follows on defining $s_k(t) = s_k$ for all $k \geq 1$ and all $t \in T$.

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