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LOCATION OF APPROXIMATIONS OF A MARKOFF THEOREM

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ABSTRACT. Relative to the first two theorems of the well known Markoff Chain (J.W.S. Cassels, "An introduction to diophantine approximation" approximations are well located. Literature is silent on the question of location of approximations in reference to the other theorems of the Chain. Here we settle it for the third theorem of the Chain.

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1. INTRODUCTION

Suppose θ is an irrational number whose simple continued fraction expansion is $[a_0, a_1, a_2, \ldots, a_n, \ldots]$. Let $\alpha_n(\theta)$ denote

$$[0, a_n, a_{n-1}, \ldots, a_1] + [a_{n+1}, a_{n+2}, \ldots]$$

Markoff Chain (Cassels [1], Kokshma [2]) is the following chain of theorems about the sequence $\{\alpha_n(\theta)\}_{n \geq 1}$:

T_1: For every irrational number θ,

$$\alpha_j(\theta) > \sqrt{5} \tag{1.1}$$

for infinity of j's and $\sqrt{5}$ cannot be increased for $\theta \sim [0, (1)^*]$.

T_2: If $\theta \not\sim [0, (1)^*]$, then

$$\alpha_j(\theta) > \sqrt{8} \tag{1.2}$$

for infinity of j's and $\sqrt{8}$ cannot be increased for $\theta \sim [0, (2)^*]$.

T_3: If $\theta \nsim [0, (1)^*]$ or $[0, (2)^*]$ then

$$\alpha_j(\theta) > (\sqrt{221})/5 \tag{1.3}$$

for infinity of j's and $(\sqrt{221})/5$ cannot be increased for $\theta \sim [0, (2,2,1,1)^*]$

T_4: If $\theta \nsim [0, (1)^*]$ or $[0, (2)^*]$ or $[0, (2,2,1,1)^*]$ then

$$\alpha_j(\theta) > (\sqrt{1517})/13 \tag{1.4}$$

for infinity and j's and $(\sqrt{1517})/13$ cannot be increased for

$$\theta \nsim [0, (2,2,1,1,1,1)^*] \text{ etc.}$$

It is know that the sequence of constants $\sqrt{5}$, $\sqrt{8}$, $(\sqrt{221})/5$, $(\sqrt{1517})/13, \ldots,$ increases to 3. So the theorems say something non-trivial about θ's in which all quotients are eventually 1 or 2 only.

As regards (1.1) and (1.2) we have an ad-hoc idea of the j's satisfying them. In reference to T_1 we know that : one j must occur in $\{n, n+1, n+2\}$ $\forall$ $n \geq 1$. Relative to T_2, we have a similar result : if $a_{n+2} = 2$ then a $j \in \{n, n+1, n+2\}$. These may be found in Wright [3] or Prasad and Lari [4].

But the literature is surprisingly silent on such results in reference to T_3, T_4, etc. In this article we announce one such result in reference to T_3 in the following theorem:

2. MAIN RESULTS

THEOREM. $a_{n+2} = 2$ and $a_{n+3} = 1$
then $\alpha_j(\theta) > (\sqrt{221})/5$ for at least one $j \in \{n, n+1, n+4\}$

REMARK. Our method gives a way to try for similar results on T_4, T_5, etc.

PROOF. Suppose $\theta = [a_0, a_1, \ldots, a_n, \ldots]$, $a_{n+2} = 2$ and $a_{n+3} = 1$

If $a_{n+1} \geq 3$; $\alpha_n(\theta) > 3$ and we are through.

If $a_{n+1} = 1$; $\alpha_{n+1}(\theta) > [0,2] + [2,2] = 3$ and we are through.

If $a_{n+1} = 2$ and $a_{n+4} \geq 2$ then $\alpha_{n+1}(\theta) > [0,3] + [2,1,1] = 3$ and we are through.

For the left out θ's: $a_{n+1} = 2 = a_{n+2}$ and $a_{n+3} = 1 = a_{n+4}$.

To deal with them, we put

$$\alpha = [0, a_n, a_{n-1}, \ldots, a_1],$$

$$\beta = [0, a_{n+5}, a_{n+6}, \ldots\ldots],$$

$$t = (\sqrt{221})/5$$

and argue over all possible values of β . We note:

$$'\alpha_{n+4}(\theta) > t' \Leftrightarrow '\alpha[5-(5t-3)\beta] > [(2t-7)\beta - 12]'$$

$$\text{So } \beta \leq \frac{12}{12t-7} \Rightarrow \alpha_{n+4}(\theta) > t$$

We next check:

$$\alpha_n(\theta) > t \Leftrightarrow \alpha > f_1(\beta) = \frac{(5t-12) + (3t-7)\beta}{(5 + 3\beta)}$$

$$\alpha_{n+1}(\theta) > t \Leftrightarrow \alpha < f_2(\beta) = \frac{(12-4t) + (7-2t)\beta}{(2t-5) - (3-t)\beta}$$

and $f_2(\beta) - f_1(\beta) = A_1 \left(\beta + \frac{251 - 5t}{146}\right) \left(\beta - \frac{5t - 9}{14}\right)$

where $A_1 = t(10-3t)\,(5+3\beta)^{-1}\,\{(2t-5) - (3-t)\beta\}^{-1}$: (> 0).

So $\beta > \frac{5t-9}{14}$

$\Rightarrow \quad f_2(\beta) > f_1(\beta)$

$\Rightarrow \quad \alpha > f_1(\beta)$ or $\alpha < f_2(\beta)$

$\Rightarrow \quad \alpha_n(\theta) > t$ or $\alpha_{n+1}(\theta) > t$.

Hence we confine attention to $\frac{12}{12t-7} < \beta \leq \frac{5t-9}{14}$.

In this case $\alpha_{n+4}(\theta) > t \Leftrightarrow \alpha > f_3(\beta) = \frac{(12t-7)\beta - 12}{5-(5t-3)\beta}$

Also $f_3(\beta) - f_2(\beta) = A_2\left(\beta + \frac{5t+19}{14}\right)\left(\beta - \frac{5t-9}{14}\right)$

where $A_2 = 2t(t-1)\,\{5-(5t-3)\beta\}^{-1} \,.\, \{(2t-5) - (3-t)\beta\}^{-1}$: (>0)

So if $\frac{12}{12t-7} < \beta < \frac{5t-9}{14}$ then $f_3(\beta) < f_2(\beta)$

which implies $\alpha < f_2(\beta)$ or $\alpha > f_3(\beta)$; equivalently

$\alpha_{n+1}(\theta) > t$ or $\alpha_{n+4}(\theta) > t$ and we are through.

Finally $\beta = \frac{5t-9}{14}$

$\Rightarrow f_1(\beta) = f_2(\beta) = f_3(\beta) = \frac{5t-9}{10}$ (an irrational number)

$\Rightarrow \alpha > f_1(\beta)$ or $\alpha < f_2(\beta)$, (α is rational)

$\Rightarrow \alpha_n(\theta) > t$ or $\alpha_{n+1}(\theta) > t$.

This completes the proof of the theorem.

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