ON PERVIN'S EXAMPLE CONCERNING THE CONNECTED-OPEN TOPOLOGY

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Abstract. Irudayanathan and Naimpally [1] introduced a topology for function spaces (called the "connected-open" topology) which has the property that the connected functions form a closed set provided that the codomain is completely normal. Pervin [2] gave an example showing that the proviso cannot be weakened to normality. The purpose of this note is to point out a lacuna in his demonstration, and to re-establish the validity of the example.

Key words and phrases. Function space, connected-open topology, complete normality.

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1. INTRODUCTION.

Let X and Y denote topological spaces, and F the set of all mappings from X to Y. For each connected subset K of X and each pair U, V of open subsets of Y denote by W(K; U, V) the subset

\[ \{ f \in F : f(K) \subseteq U \cup V, f(K) \cap U \neq \emptyset \neq f(K) \cap V \} \]

of F. The collection S of all these sets W(K; U, V) is a subbase for the connected-open topology T on F, introduced by Irudayanathan and Naimpally in [1] where it is proved that the collection C^-2 of all connected (Darboux) functions is T-closed if Y is completely normal.

To show that normality of Y is not sufficient for this result, Pervin [2] took Y as a modification of the Tychonoff plank, with an open interval of reals interpolated between each ordinal and its successor in the construction; appealed to cardinality to obtain a function f from the unit interval X = [0,1] onto a subset A^* \cup B^* of Y, where A^* and B^* were separated but had no disjoint neighbourhoods in Y, and where f^(-1)(y) was dense in X for every y in A^* \cup B^*; and proved that any member W(K; U, V) of S which contained f must also contain a connected function. However, this does not suffice to establish that the (non-connected) function f belongs to the T-closure of C^-2, it being perfectly possible for every subbasic neighbourhood
of a point to intersect a set without every basic neighbourhood doing so. We shall show that \( f \) is, nevertheless, a limit of connected (indeed, of continuous) functions.

2. PERVIN'S EXAMPLE REVISITED.

Let \( J \) denote the connected, compact \( T_2 \) space formed from the second uncountable ordinal \( \omega_2 \) by interpolating a copy of \( (0,1) \) between each element (other than the maximum) and its successor, and imposing the order topology on the resulting chain; and consider the product space \( Y = J \times [0,1] \). (The space \( \omega_2^\omega \) used here by Pervin instead of \([0,1]\) is homeomorphic to \([0,1]\).) Denote by \( a \) and \( b \) (respectively) the least and greatest elements of \( J \), and by \( A^* \) and \( B^* \) the following subsets of \( Y \):

\[
A^* = [a,b) \times \{1\}, \quad B^* = \{b\} \times [0,1).
\]

(Pervin's definition of these sets is incompatible with his assertion that they are connected; the above is presumably what was intended.) Considerations of cardinality establish the existence of a mapping \( f \) from \([0,1]\) onto \( A^* \cup B^* \) such that the preimage of each singleton is dense. It will now be shown that every neighbourhood of \( f \) contains a connected function.

Consider a typical basic \( T \)-neighbourhood

\[
G = \cap \{ W(K_i; U_i, V_i) : i = 1, 2, \ldots, n \}
\]

of \( f \), where (for each \( i \)) \( K_i \) is a connected subset of \([0,1]\), \( U_i \) and \( V_i \) are open in \( Y \), and \( f(K_i) \) is contained in the union of \( U_i \) and \( V_i \) and meets them both. No loss of generality will be incurred by assuming that the sets \( K_i \) are distinct, since

\[
W(K; U, V) \cap W(K; U', V') \supseteq W(K; U \cap U', V \cap V').
\]

Denoting by \( j \) the number of degenerate intervals amongst the \( K_i \), where \( 0 \leq j \leq n \), we can arrange the labelling so that \( K_i \) is a singleton for \( i \leq j \) and is non-degenerate for \( i > j \). The strategy of the proof is to determine a subset \( Z \) of \( Y \) of the form suggested by a graph in the diagram below (which see), where \( x \) is chosen to ensure that \( Z \) is contained in \( U_i \cup V_i \) for all \( i > j \), and \( z \) is selected so that \( Z \) includes at least one point of \( U_i \cap V_i \) for each \( i \); a path-connectedness argument within \( Z \) will then produce a continuous function belonging to \( G \).

For \( i > j \), \( f(K_i) \) is the whole of \( A^* \cup B^* \) and is contained in \( U_i \cup V_i \). Thus for each positive integer \( n \) the product of compact sets

\[
\{b\} \times [0,1-2^{-n}]
\]

is contained in \( U_i \cup V_i \), and a lemma of A.D. Wallace (see [3], p.142) allows us to find \( x_{i,n} \in [a,b] \) such that

\[
(x_{i,n}, b] \times [0,1-2^{-n}] \subseteq U_i \cup V_i.
\]
Now \([a,b)\) inherits from its cofinal subset \(W_\Omega \setminus \{b\}\) the property that each countable subset is bounded above: choosing then a strict upper bound \(x_i < b\) for the sequence \((x_i, n)\) we see that

\[
[x_i, b] \times [0,1) \subseteq U_i \cup V_i;
\]

and so if \(x\) denotes the maximum of the elements \(x_i\) here chosen, we have

\[
[x, b] \times [0,1) \subseteq U_i \cup V_i \quad \text{for all } i > j.
\] (2.1)

(In the event that \(j = n\), i.e. that all the \(K_i\) are degenerate, (2.1) may be obtained by an arbitrary choice of \(x < b\).)

Still considering the case \(i > j\), we see from (2.1) that the connected set \(A^* \cup (x, b] \times [0,1)\) is contained in the union of \(U_i\) and \(V_i\) and intersects them both; so it must be possible to choose a point \(t(i) = (t(i)_1, t(i)_2)\) of \(U_i \cap V_i\) such that either \(t(i) \in A^*\), or else \(t(i) \in (x, b] \times [0,1)\): and in the latter case, the observations that \(U_i \cap V_i\) is a neighbourhood of \(t(i)\) and that \(b\) is not isolated in \(J\) will allow us to assume that \(x < t(i)_1 < b\). Turning now to the case \(i \leq j\), \(f(K_i)\) is here a single point of \((A^* \cup B^*) \cap U_i \cap V_i\); if this point lies in \(A^*\) we denote it...
by \( t(i) = (t(i)_1, t(i)_2) \), while if it belongs to \( B^* \), an argument like that above will yield \( t(i) = (t(i)_1, t(i)_2) \) in \( U_i \cap V_i \) satisfying \( x < t(i)_1 < b \). Lastly let \( z \) denote the maximum of \( t(1)_1, t(2)_1, \ldots, t(n)_1 \): the consequence of the choices of \( x \) and of \( z \) is that the set

\[
Z = [a, z] \times \{ 1 \} \cup [x, z] \{ 0, 1 \}
\]

includes the point \( t(i) \) of \( U_i \cap V_i \) for every \( i \), and is contained in \( U_i \cup V_i \) for those values of \( i \) (if any) for which \( K_i \) is non-degenerate.

Now since \( z < b \), the interval \((x, z)\) contains only countably many elements of \( \bar{W}_\Omega \) and only countably many interpolated real intervals or parts thereof, so it possesses a countable dense subset. It is routine to verify that it contains a supremum and an infimum for each of its bounded subsets, and it has no least nor greatest element and no gaps. Thus by a well-known characterization due to Hausdorff ([4], p. 54) it is homeomorphic to the real line. Then \([x, z]\) and, similarly, \([a, z]\) are homeomorphic to bounded closed real intervals; and \( Z \), being essentially the unit square in the real plane with a line segment attached to one corner, is path-connected. Choosing distinct elements \( k_i \) in \( K_i \) for each \( i \), which will be possible since the \( K_i \) are themselves distinct intervals, this guarantees the existence of a continuous (and therefore connected) function \( g : [0, 1] \rightarrow Z \) such that \( g(k_i) = t(i) \) for each \( i \). Regarding \( g \) as a mapping into \( Y \), we see that it is common to all the sets \( W(K_i; U_i, V_i) \) and the demonstration is complete.

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