RING HOMOMORPHISMS ON H(G)

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ABSTRACT. It is shown that a ring homomorphism on H(G), the algebra of analytic functions on a regular region G in the complex plane, is either linear or conjugate linear provided that the ring homomorphism takes the identity function into a nonconstant function.

KEY WORDS AND PHRASES. Ring homomorphism, Algebra of analytic function, Linear, Conjugate linear.

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1. INTRODUCTION.

An operator M on a commutative algebra A is called a ring homomorphism if for all x, y ∈ A, M(x + y) = M(x) + M(y) and M(xy) = M(x)M(y). Throughout this paper G denotes a region, i.e., a connected open set in the complex plane, H(G) denotes the algebra of analytic functions on a region G in the complex plane equipped with the topology of uniform convergence on compact subsets of G, M denotes a nontrivial ring homomorphism on H(G), and I denotes the identity function on G. A region G in C is called regular if G = interior (closure G). The rationals, reals and complex numbers are denoted by Q, R, and C respectively.

If N is a maximal ideal in H(G) then the quotient algebra H(G)/N is isomorphic (as an algebra) to C if and only if N is the kernel of a linear homomorphism. Henriksen [1] has shown that H(G)/N is isomorphic (as a ring) to C where G = C and the maximal ideal M is not closed. This implies that there exist discontinuous homomorphisms from the ring of entire functions onto C.

In this paper we show that if G is a regular region in C and a ring homomorphism M on H(G) takes the identity function I to a non-constant function, then M is necessarily continuous. Essentially we prove that homomorphisms under consideration preserve constants. However, the results of this fact can be obtained by the techniques used in [2] and [3] except in the case G = C.
If $M$ is a ring homomorphism on $H(G)$ then the following assertions are equivalent:

1) $M$ is continuous,

2) either $M(k) = k$ for all $k \in \mathbb{C}$ or $M(k) = \overline{k}$ for all $k \in \mathbb{C}$,

3) $M$ is either linear or conjugate linear,

4) there exists $h \in H(G)$ with $h(G) \subseteq G$ such that $M(f) = foh$ for all $f \in H(G)$ or there exists $h \in H(G)$ with $\overline{h(G)} \subseteq G$ such that $M(f) = foh$ for all $f \in H(G)$.

The implications $4 \implies 1 \implies 2 \implies 3$ are trivial or easy to prove; $3 \implies 4$ is the content of Lemma 2.1.

We show that a ring homomorphism $M$ on $H(G)$ which takes the identity function to a non-constant function is necessarily linear or conjugate linear using Nienhuys-Thiemann’s theorem [4] which states that given any two countable dense subsets $A$ and $B$ of $\mathbb{R}$ there exists an entire function which is real valued and increasing on the real line $\mathbb{R}$ such that $f(A) = B$. In Section 2 we give some lemmas and a theorem of Nienhuys and Thiemann. In Section 3 we prove the following main theorem.

THEOREM 1.1. Let $G$ be a regular region in $\mathbb{C}$ and let $M$ be a ring homomorphism on $H(G)$ such that $M(I)$ is not a constant function where $I$ is the identity function. Then $M(i) \neq \pm i$. Further

a) if $M(i) = i$ then $M$ is linear,

b) if $M(i) = -i$ then $M$ is conjugate linear,

2. LEMMAS.

The following lemma is well known and we give the proof for the sake of completeness.

LEMMA 2.1. Let $M$ be a ring homomorphism on $H(G)$. If $M$ is linear then there exists $h \in H(G)$ with $h(G) \subseteq G$ such that $M(f) = foh$ for all $f \in H(G)$.

PROOF. Let $M(I) = h$ and $z_0 \in G$. We claim that $h(z_0) \in G$. Suppose not, then

\[ (I - h(z_0)) \left( \frac{1}{I - h(z_0)} \right) = 1. \]

Applying $M$ on both sides and evaluating at $z_0$ with the observation that $M(h(z_0)) = h(z_0)$ we obtain

\[ 0 = (M(I)(z_0) - h(z_0)) \frac{1}{I - h(z_0)} (z_0) \]

\[ = M(I - h(z_0))(z_0) \frac{1}{I - h(z_0)} (z_0) \]

\[ = M(I)(z_0) \]

\[ = 1, \]

which is a contradiction. Since $z_0$ is arbitrary we have $h(G) \subseteq G$.

Since $h(z_0) \in G$ we have $f - f(h(z_0)) \in H(G)$ and

\[ f - f(h(z_0)) = (I - h(z_0)) \frac{f - f(h(z_0))}{I - h(z_0)}. \]

Applying $M$ on both sides and evaluating at $z_0$ we obtain
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\[ M(f)(z_0) = M(f(h(z_0)))(z_0) = f(h(z_0)) \]

Since \( z_0 \) is arbitrary the result follows.

**Lemma 2.2.** Let \( G \) be a regular region in \( \mathbb{C} \) and \( M \) be a ring homomorphism on \( H(G) \) with \( M(1) = i \). If \( M(I) = h \) is not a constant function then \( h(G) \subseteq G \).

**Proof.** Since \( M \) is a nontrivial ring homomorphism it is easy to show that \( M(\alpha) = \alpha \) for all \( \alpha \in \mathbb{Q} \). Since \( M(1) = i \) we have \( M(\alpha + i\beta) = \alpha + i\beta \) where \( \alpha, \beta \in \mathbb{Q} \). Let \( z_0 \in G \) such that \( h(z_0) \in \mathbb{Q} + i\mathbb{Q} \). Just as in the above lemma it is easy to show that \( h(z_0) \in G \). Since \( h \) is not a constant function we have \( h(z_0) \in G \) for a dense set of \( z_0 \) in \( G \) and since \( h(G) \) is open we have \( h(G) \subseteq \text{interior}(\text{closure } G) = G \).

Let \( K \in \mathbb{Q} \). Denote by \( H_K \) the set of all entire functions which map \( \mathbb{Q} + ik \) into \( \mathbb{Q} \) except possibly for one point of \( \mathbb{Q} + ik \) and also denote by \( EM \) the class of entire functions whose restriction to \( \mathbb{R} \) is a real monotonically increasing function. The proof of Lemma 2.3 follows the proof of the following theorem of Nienhuys & Thiemann [4].

**Theorem 2.1.** Let \( S \) and \( T \) be countable everywhere dense subsets of \( \mathbb{R} \), let \( p \) be a continuous positive real function such that \( \lim_{t \to t^+} t^{-n}p(t) = 0 \) for all \( n \in \mathbb{N} \) and let \( f_0 \in EM \).

Then there exists a function \( f \in EM \) such that

1. \( f \) is strictly increasing on \( \mathbb{R} \) and \( f(S) = T \),
2. \( |f(z) - f_0(z)| < p(|z|) \) for all \( z \in \mathbb{C} \).

**Lemma 2.3.** Let \( k \in \mathbb{Q}, \beta \in \mathbb{R} \) and \( \alpha \in \mathbb{Q} + ik \). Then there exists an entire function \( f \in H_K \) such that \( f(\alpha) = \beta \) and \( f(\mathbb{Q} + ik) = \{\beta\} \cup \mathbb{Q} \).

**Proof.** In Nienhuys and Thiemann's Theorem [4] take \( S = \mathbb{Q} \) and \( T = \{\beta\} \cup \mathbb{Q} \). Let \( x_1, x_2, \ldots \) be an enumeration of \( \mathbb{Q} \). Then as in the proof of this theorem there exists an entire function \( g \) such that \( g(x_1) = \beta \) and \( g(\mathbb{Q}) = \{\beta\} \cup \mathbb{Q} \). Let \( x_1 = \alpha - ik \) and \( h(z) = z - ik \). Then \( f = gh \) is the desired function.

3. **Proof of the Main Theorem.**

It is easy to see that \( M \) is linear over the field of rational numbers and hence we have \( -1 = M(-1) = M(i^2) = M(i)^2 \) which implies \( M(i) = \pm i \). We prove here only Part a) of the theorem and the proof of Part b) follows similarly.

Since \( h = M(I) \) is a nonconstant analytic function on \( G \), \( h(G) \) is a nonempty open set in \( \mathbb{C} \) and by Lemma 2.2, \( h(G) \subseteq G \). Hence there exists \( k \in \mathbb{Q} \) such that \( S = (\mathbb{R} + ik) \cap h(G) \) has an interval parallel to real axis. Let \( f \in H(G) \) and \( h(z_0) \in (\mathbb{Q} + ik) \cap G \). Then applying \( M \) on both sides and evaluating at \( z_0 \) in the following

\[ f - f(h(z_0)) = (I - h(z_0)) \left( \frac{f - f(h(z_0))}{I - h(z_0)} \right) \]

we obtain

\[ M(f - f(h(z_0)))(z_0) = 0 \]

for all \( z_0 \) in \( G \) such that \( h(z_0) \in \mathbb{Q} + ik \). Thus we have
\[ M(f)(z_0) = M(f(h(z_0)))(z_0), \text{ for all } f \in H(G) \text{ and for all } h(z_0) \in Q + ik. \]  
(1)

Since a function \( f \) in \( H_k \) takes \( Q + ik \) into rationals except for one point of \( Q + ik \), we obtain \( M(f(h(z_0))) = f(h(z_0)) \) whenever \( h(z_0) \) is in \( (Q + ik) \cap G \). Since \( f \) is analytic we obtain
\[ M(f) = fo h, \text{ for all } f \in H_k. \]  
(2)

For a given \( \beta \in \mathbb{R} \) and each \( h(z_0) \) in \( Q + ik \), by Lemma 2.3 there exists an entire function in \( H_k \) such that \( f(h(z_0)) = \beta \). Substituting this in (1) on the one hand we obtain
\[ M(f)(z_0) = M(\beta)(z_0) \]
and evaluating (2) at \( z_0 \) on the other we obtain
\[ M(f)(z_0) = fo h(z_0) = f(h(z_0)) = \beta \]

Thus we obtain from the above two relations that
\[ M(\beta)(z_0) = \beta \text{ for all } z_0 \in h^{-1} (Q + ik) \cap G. \]

Since \( M(\beta) \) is analytic we have \( M(\beta) = \beta \). Thus we have \( M(k) = k \) for all \( k \in \mathbb{C} \). This implies \( M \) is linear.

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