ABELIAN THEOREM FOR THE STIELTJES TRANSFORM OF DISTRIBUTIONS

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ABSTRACT. First, a definition is given of the Stieltjes transform of distributions which contains some well-known. Then, a structural theorem for distributions having S-asymptotic is proved. This makes possible to prove two theorems of the Abelian type valued for Stieltjes transforms of distributions given in different ways.

KEYS WORDS AND PHRASES. Stieltjes transform, S-asymptotic, Abelian type theorem, distribution, generalized function.

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1. INTRODUCTION.

It is possible to define the Stieltjes transform of distributions in various ways. One of them is the so-called direct approach in which we construct a basic space $A \supset \mathcal{D}$ of smooth functions to which the set $\{(s+t)^{-r-1}, \operatorname{Im} s \neq 0, r \geq 0\}$ belongs. Then, we define a transform for T , belonging to the dual space A' , by the expression $\langle T(t), (s+t)^{-r-1} \rangle$. In this paper we shall deal only with such definitions. If the constructed space A' is such that $T \in A'$ has the support belonging to $[0, \infty)$, we have the classical Stieltjes transform, extended to generalized functions in one dimension.

We shall give a definition of the Stieltjes transform in many-dimensional case not only to have a new generalisation of the Stieltjes transform, but to prove Abelian type theorems valued for the Stieltjes transforms given in different ways.

2. NOTATIONS AND DEFINITIONS.

N is the set of natural numbers, $N_0 = N \cup \{0\}$. If $a, b \in \mathbb{R}^n$, $z \in \mathbb{C}^n$, $z = (x_1 + iy_1, \dots, x_n + iy_n)$ and $e = (1, \dots, 1)$, then: $(a \cdot b) = \sum_{i=1}^n a_i b_i$; $z^a = \prod_{i=1}^n (x_i + iy_i)^{a_i}$; $|z^a| = \prod_{i=1}^n |x_i + iy_i|^{a_i}$; $z^e = \prod_{i=1}^n z_i$; $\|a\|^2 = (a \cdot a)$; $a \geq 0$ means $a_i \geq 0$ for all $i = 1, \dots, n$.

Γ will be a convex acute cone with vertex at zero; $B(0, r)$ the closed ball with the center at zero and with radius $r > 0$; $S(0, 1)$ the unit sphere, all in $\mathbb{R}^n$. $\overset{\circ}{\Gamma}$ is the interior of Γ .

DEFINITION 1. Let $h_1, h_2 \in \Gamma$. We say that $h_1 \geq h_2$ in Γ if $h_1 \in h_2 + \Gamma$. For a complex valued function $G(h)$, $h \in \Gamma$, $\lim_{h \in \Gamma, h \rightarrow \infty} G(h) = A \in \mathbb{C}$, $h \in \Gamma$, $h \rightarrow \infty$ if for every $\varepsilon > 0$ there exists $h(\varepsilon) \in \Gamma$ such that $|G(h) - A| < \varepsilon$ when $h \geq h(\varepsilon)$ in Γ .

If $\overset{\circ}{\Gamma} \neq \emptyset$, we can prove the following Lemma:

LEMMA 1. Suppose $\overset{\circ}{\Gamma} \neq \emptyset$ and $B(a, r) \subset \overset{\circ}{\Gamma}$. We denote by $\Gamma_1 = \bigcup_{\lambda > 0} \lambda B(a, r)$. If $\lim_{h \in \Gamma, h \rightarrow \infty} G(h) = A$, then $\lim_{h \in \Gamma_1, h \rightarrow \infty} G(x+h) = A$ for every $x \in \mathbb{R}^n$.

PROOF. By Definition 1 for every $\varepsilon > 0$ there exists $h(\varepsilon)$ such that $|G(h) - A| < \varepsilon$, $h \in h(\varepsilon) + \Gamma$. Now $x - h(\varepsilon) + \beta_0 a \in B(\beta_0 a, \beta_0 r)$ if $\beta_0 \geq \|x - h(\varepsilon)\|/r$, β_0 depends on ε . Hence $x - h(\varepsilon) + \beta_0 a \in \Gamma_1 \subset \Gamma$. Since Γ is a convex cone, $x - h(\varepsilon) + \beta_0 a + \Gamma_1 \subset \Gamma + \Gamma_1 \subset \Gamma$ and $x + \beta_0 a + \Gamma_1 \subset h(\varepsilon) + \Gamma$. It follows that $|G(x+h) - A| < \varepsilon$ for $h \in \beta_0 a + \Gamma_1$.

REMARK. In Lemma 1 we can choose the ball $B(a, r)$ in such a way that the distance between $B(a, r)$ and any coordinate axis X_i , $i = 1, \dots, n$, in $\mathbb{R}^n$ is positive, $d(B(a, r), X_i) \geq \alpha > 0$. Then $|a_i| - r \geq \alpha > 0$, $i = 1, \dots, n$. We denote by $\Gamma_2 = \bigcup_{\lambda > 0} \lambda B(a, r)$, for such constructed ball $B(a, r)$. If $u \in h_0 + \Gamma_2$ for a $h_0 \in \Gamma_2$, then $u = \lambda_0 a + \lambda_0 z + \lambda a + \lambda y$, where $z, y \in B(0, r)$; $\lambda_0, \lambda > 0$. It is easy to see that $|u_i| \geq (\lambda + \lambda_0)(|a_i| - r) \geq (\lambda + \lambda_0)\alpha$ and for any $M > 0$ we can find h_0 with the property $|u_i| \geq M$. Hence, if $h \rightarrow \infty$, $h \in \Gamma_2$, then $|h_i| \rightarrow \infty$, $i = 1, \dots, n$.

By $\mathcal{P}$ we denote the set of all the real and positive functions c . Notations for the spaces of distributions are as in the book of L. Schwartz [1].

DEFINITION 2. A distribution $T \in \mathcal{D}'$ has the S-asymptotic in Γ related to the function $c \in \mathcal{P}$ and with the limit $U \in \mathcal{D}'$, if the following limit exists

$$\lim_{h \in \Gamma, h \rightarrow \infty} \langle T(x+h)/c(h), \phi(x) \rangle = \langle U, \phi \rangle, \quad \phi \in \mathcal{D}. \quad (2.1)$$

Then we write $T(x+h) \sim c(h) \cdot U(x)$, $h \in \Gamma$ and we say that T has the S-asymptotic in $\mathcal{D}'$ (Pilipović and Stanković [2] and Stanković [3]).

We shall use the same definition for a $T \in \mathcal{B}' \equiv \mathcal{D}'(L^\infty)$ and $\phi \in \mathcal{D}(L^1)$ stressing that T has the S-asymptotic in $\mathcal{B}'$.

If the interior of the cone Γ is not empty, $\overset{\circ}{\Gamma} \neq \emptyset$, then we can give the analytical expression of the limit distribution U :

Suppose that $T \in \mathcal{D}'$ and has the S-asymptotic with the limit $U \neq 0$. Then there exists a $\phi_0 \in \mathcal{D}$ such that $\langle U, \phi_0 \rangle \neq 0$. For this ϕ_0 and $t \in \mathbb{R}^n$, using Lemma 1, we have

$$\lim_{h \in \Gamma_2, h \rightarrow \infty} \frac{c(h+t)}{c(h)} \langle \frac{T(x+(h+t))}{c(h+t)}, \phi_0(x) \rangle = \lim_{h \in \Gamma_2, h \rightarrow \infty} \langle \frac{T((x+t)+h)}{c(h)}, \phi_0(x) \rangle. \quad (2.2)$$

From this relation it follows the existence of the following limit

$$\lim_{h \in \Gamma_2, h \rightarrow \infty} c(h+t)/c(h) = d(t), \quad t \in \mathbb{R}^n \quad (2.3)$$

and that U satisfies the equation

$$d(t) \langle U, \phi_0 \rangle = \langle U(x+t), \phi_0(x) \rangle, \quad d(0) = 1, \quad t \in \mathbb{R}^n. \quad (2.4)$$

Since U , as a distribution, has all the derivatives, it follows from relation (2.4) that:

$$\begin{aligned} [d(t+\Delta t_1) - d(t)] \langle U, \phi_0 \rangle &= \langle U(x+t+\Delta t_1) - U(x+t), \phi_0(x) \rangle \\ &= [d(\Delta t_1) - d(0)]d(t) \langle U, \phi_0 \rangle, \quad d(0) = 1. \end{aligned} \quad (2.5)$$

It is now easy to prove that (Pilipović and Stanković [2]):

$$d(t) = \exp(a \cdot t) \quad \text{and} \quad U(t) = C \exp(a \cdot t), \quad t \in \mathbb{R}^n. \quad (2.6)$$

3. STIELTJES TRANSFORM OF DISTRIBUTIONS

In the following we shall use the well-known function $\eta_\omega \in C^\infty$, $\omega > 0$ (Vladimirov [4]):

$$\eta_\omega(x) = \int_{B(0,2\omega)} q_\omega(x-t) dt, \quad x \in \mathbb{R}^n \quad (3.1)$$

where

$$q_\omega(x) = \begin{cases} D\omega^{-n} \exp\left(-\frac{\omega^2}{\omega^2 - \|x\|^2}\right), & \|x\| < \omega \\ 0, & \|x\| \geq \omega \end{cases}; \quad D \int_{\mathbb{R}^n} q_1(t) dt = 1. \quad (3.2)$$

The function η_ω , has the properties: $0 \leq \eta_\omega(x) \leq 1$, $x \in \mathbb{R}^n$; $\eta_\omega(x) = 1$, $x \in B(0, \omega)$; $\eta_\omega(x) = 0$, $\|x\| > 3\omega$; $|D^k \eta_\omega(x)| \leq C_k \omega^{-(k \cdot e)}$, $x \in \mathbb{R}^n$. The constants C_k do not depend on ω .

DEFINITION 3. The Stieltjes transform of a distribution $T \in \mathcal{D}'$ (S_ρ -transform) is defined by the limit

$$\lim_{\omega \rightarrow \infty} \langle T(x), \eta_\omega(x)(s+x)^{-(\rho+e)} \rangle = S_\rho(T)(s), \quad s \in (\mathbb{C} \setminus \mathbb{R})^n, \quad (3.3)$$

if it exists for a $\rho \in \mathbb{R}^n$. By e we denoted $e = (1, \dots, 1)$.

REMARK. s can belong to a larger set, as well. This depends on the support of T . So $S_\rho(\delta)(s) = s^{-(\rho+e)}$, $s \neq 0$.

We shall give a relation between Definition 3 of the Stieltjes transform and definitions of some other authors frequently used in many papers. All of them are in one dimension. Let us start with the classical definition.

If T is defined by the function f , $\text{supp } f \subset [0, \infty)$, Definition 3 gives the classical Stieltjes transform, if it exists. Let $s \in (\mathbb{C} \setminus (-\infty, 0])$, then

$$S_\rho(f)(s) = \lim_{\omega \rightarrow \infty} \int_0^{3\omega} f(t) \eta_\omega(t)(s+t)^{-(\rho+1)} dt = \lim_{\omega \rightarrow \infty} \int_0^\omega f(t)(s+t)^{-(\rho+1)} dt + \\ + \lim_{\omega \rightarrow \infty} \int_\omega^{3\omega} f(t) \eta_\omega(t)(s+t)^{-(\rho+1)} dt. \quad (3.4)$$

We have only to prove that:

$$\lim_{\omega \rightarrow \infty} \int_0^{3\omega} f(t) \eta_\omega(t)(s+t)^{-(\rho+1)} dt = 0, \quad s \in (\mathbb{C} \setminus (-\infty, 0]) \quad (3.5)$$

when the classical Stieltjes transform exists.

Since for $\omega \leq x \leq 3\omega$

$$\eta_\omega(x) = \int_{-2\omega}^{2\omega} q_\omega(x-t) dt = \int_{(x-2\omega)/\omega}^1 q_1(y) dy; \quad (3.6)$$

the function $\eta_\omega(x)$ is positive and monotone decreasing in this interval. By the mean value theorem, there exists a ξ , $0 < \xi < 2$ such that

$$\int_\omega^{3\omega} \eta_\omega(t) \text{Re}[f(t)(s+t)^{-(\rho+1)}] dt = \eta_\omega(\omega) \int_\omega^{\omega+\xi\omega} \text{Re}[f(t)(s+t)^{-(\rho+1)}] dt. \quad (3.7)$$

The last integral tends to zero when $\omega \rightarrow \infty$, because we supposed that the Stieltjes transform of function f exists. We have the same situation with the imaginary part of the

integral from relation (3.5).

Lavoine and Misra [5-6] defined the Stieltjes transform of distributions belonging to a subset $J'(r), r \geq -1$ of $\mathcal{D}'$, which is used in many papers. A distribution T belongs to $J'(r)$ if and only if there exist $m \in \mathbb{N}$ such that $T = D^m G$, where G is a locally integrable function having a support in $[0, \infty)$ and $G(x) = o(x^{r+m-\alpha})$, $\alpha > 0$. The Stieltjes transform of $T \in J'(r)$ is, by these two authors:

$$S_r(T) = (r+1) \dots (r+m) \int_0^\infty G(t)(s+t)^{-r-m-1} dt, \quad s \in \mathbb{C} \setminus (-\infty, 0]. \quad (3.8)$$

For a $T \in J'(r)$, $r = \rho$, relation (3.3) gives

$$\begin{aligned} S_\rho(T)(s) &= \lim_{\omega \rightarrow \infty} \langle T(x), \eta_\omega(x)(s+x)^{-(\rho+1)} \rangle = \\ &= (-1)^m \int_{k=1}^m \binom{m}{k} (\rho+1) \dots (\rho+m) \lim_{\omega \rightarrow \infty} \int_0^\infty \frac{G(x)}{(s+x)^{\rho+k+1}} D^{m-k} \eta_\omega(x) dx. \end{aligned} \quad (3.9)$$

We have two types of integrals

$$\int_0^\infty \frac{G(x)}{(s+x)^{\rho+m+1}} \eta_\omega(x) dx \quad \text{and} \quad \int_\omega^{3\omega} \frac{G(x)}{(s+x)^{\rho+k+1}} D^{m-k} \eta_\omega(x) dx, \quad m-k \geq 1. \quad (3.10)$$

For the first one we proceed as in the case of the classical Stieltjes integral.

The second one tends to zero when $\omega \rightarrow \infty$. We have only to start from

$$\int_\omega^{3\omega} \frac{G(x)}{(s+x)^{\rho+m+1}} (s+x)^{m-k} D^{m-k} \eta_\omega(x) dx \quad (3.11)$$

and to use the integration by parts.

In the next cases the authors followed the construction of the Stieltjes transform as we mentioned in the Introduction. The basic spaces A are topological vector spaces of complex valued smooth functions. The space A contains $\mathcal{D}$ and the topology of $\mathcal{D}$ is stronger than that induced on it by A . The restriction T of an element $\tilde{T} \in A'$ to $\mathcal{D}$ is in $\mathcal{D}'$. For a $\rho \geq 0$ and s belonging to a subset of $\mathbb{C}$, $(s+t)^{-(\rho+e)}$ is in A . The Stieltjes transform of a $\tilde{T} \in A'$ is defined by $\langle \tilde{T}(t), (s+t)^{-(\rho+e)} \rangle$.

In all the cases, we shall list, the family $\{(s+t)^{-(\rho+e)} \eta_\omega(t), \omega > 0\}$ converges in A to $(s+t)^{-(\rho+e)}$ when $\omega \rightarrow \infty$. Then for a $\tilde{T} \in A'$ and $T \in \mathcal{D}'$, we have:

$$S_\rho(T)(s) = \lim_{\omega \rightarrow \infty} \langle \tilde{T}(x), \eta_\omega(x) (s+x)^{-(\rho+e)} \rangle = \langle \tilde{T}(x), (s+x)^{-(\rho+e)} \rangle \quad (3.12)$$

By Zemanian [7] it is $\rho = 0$ and the basic space A is $J_{c,d} = \{\psi \in C_{(0,\infty)}^\infty, P_{c,d,k}(\psi) = \sup_{0 < t < \infty} \chi_{c,d}(t) |(tD)^k \psi(t)| < \infty, k \in N_0\}$, where $\chi_{c,d}(t) = t^c$, $1 \leq t < \infty$; $\chi_{c,d}(t) = t^d$, $0 < t < 1$ and $c < 1/2$, $d > -1/2$. The topology of $J_{c,d}$ is defined by the seminorms $P_{c,d,k}$, $k \in N_0$.

By Pandey [8] the basic space A is $S_\alpha = \{\phi \in C_{(0,\infty)}^\infty, \gamma_k(\phi) = \sup (1+x^\alpha) |(xD)^k \phi(x)| < \infty, k \in N_0\}$. The topology in S_α is defined by the seminorms γ_k , $k \in N_0$. The function $(s+t)^{-(\rho+1)} \in S_\alpha$ for $\alpha \leq \rho+1$, but the family $\{(s+t)^{-(\rho+1)} \eta_\omega(t), \omega > 0\}$ tends to $(s+t)^{-(\rho+1)}$ in S_α for $\alpha < \rho+1$ when $\omega \rightarrow \infty$.

Bremermann [9] introduced the basic space $0_\alpha = \{\zeta \in E(R), \zeta^{(k)}(t) = 0(t^\alpha), k \in N_0\}$. The topology of 0_α is that induced by E . He treated the case $\rho = 0$, $\text{Im } s \neq 0$ and $\alpha \geq -1$. For the Stieltjes transform of distributions in many dimensional case see also [10].

4. DISTRIBUTIONS HAVING THE S-ASYMPTOTIC

We shall prove a structural theorem for the distributions having the S-asymptotic in a cone Γ with the nonempty interior, $\Gamma \neq \emptyset$.

THEOREM 1. Suppose $T_0 \in \mathcal{B}'$ and $T_0(x+h) \stackrel{S}{\sim} 1 \cdot U(x)$, $h \in \Gamma$ in $\mathcal{D}'$, then

- $U = C$;
- $T_0 = \sum_{i=0}^2 \Delta^{ik} F_i$, where F_i are continuous functions belonging to L^∞ ;
- For every $0 \leq i \leq 2$ functions $F_i(x+h)$ converge uniformly to a constant when x belongs to a compact set K and $h \in \Gamma$, $h \rightarrow \infty$;
- T_0 has the S-asymptotic in $\mathcal{B}'$, as well, related to $c = 1$ and with the limit $U = C$ in the cone Γ .

PROOF. a) By relation (2.6) U has to be a constant because $c = 1$.

b) From the fact $T_0 \in \mathcal{B}'$ it follows that $(T_0 * \hat{\zeta}) \in L^\infty$ for every $\zeta \in \mathcal{D}$ (Schwartz [1], II, p. 57) and the set of distributions $\mathcal{Q} = \{T_h \equiv T_0(x+h), h \in \mathbb{R}^n\}$ is weakly bounded and bounded in $\mathcal{D}'$.

In addition to $\mathcal{Q}$, we shall construct an other bounded set of distributions. We denote by $S = \{\psi \in \mathcal{D}, \|\psi\|_{L^1} \leq 1\}$. We have seen that for a fixed $\zeta \in \mathcal{D}$, $(T_0 * \hat{\zeta}) \in L^\infty$. Now, for every $\psi \in S$:

$$\begin{aligned} |\langle T_0 * \hat{\psi}, \zeta \rangle| &= |\langle T_0 * \hat{\zeta}, \psi \rangle| = \left| \int_{\mathbb{R}^n} (T_0 * \hat{\zeta})(t) \psi(t) dt \right| \leq \\ &\leq \|T_0 * \hat{\zeta}\|_{L^\infty} \|\psi\|_{L^1}. \end{aligned} \quad (4.1)$$

Hence the set of regular distributions, defined by the set of continuous functions $H = \{U_\psi \equiv T_0 * \psi, \psi \in S\}$ is weakly bounded and bounded in $\mathcal{D}'$.

A set $\mathcal{W}' \in \mathcal{D}'$ is bounded if and only if for every $\alpha \in \mathcal{D}$ the set of functions $\{T * \alpha, T \in \mathcal{W}'\}$ is bounded on every compact set M belonging to $\mathbb{R}^n$ (Schwartz [1], II, p. 50). Hence $\{T * \alpha, T \in \mathcal{W}'\}$ defines a bounded set of regular distributions. In such a way $\{T_h * \zeta, T_h \in \mathcal{Q}\}$ and $\{U_\psi * \zeta, U_\psi \in H\}$ give two bounded sets of regular distributions. Now, for these two sets we can repeat twice a part of the proof of Theorem XXII from Schwartz [1], II, p. 51.

We denote by Ω an open neighbourhood of zero in $\mathbb{R}^n$ which is relatively compact in $\mathbb{R}^n$, $\text{cl}\Omega = K$ a compact set. Then, by the mentioned part of the proof by Schwartz [1], there exist $m_1 \geq 0$ and $m_2 \geq 0$, such that the mappings $(\alpha, \beta) \rightarrow U_\psi * (\alpha * \beta)$ or $(\alpha, \beta) \rightarrow T_h * (\alpha * \beta)$ are equicontinuous and map $\mathcal{D}_\Omega^{m_1} \times \mathcal{D}_\Omega^{m_1}$ or $\mathcal{D}_\Omega^{m_2} \times \mathcal{D}_\Omega^{m_2}$ into L_B^∞ ; B is the ball $B(0, r)$ where r is a positive number. Hence, for every $x \in B$ and $h \in \mathbb{R}^n$ the function $(T_h * \alpha * \beta)(x) = (T_0 * \alpha * \beta)(x+h)$ is continuous.

Let $Z(0, \rho)$ be a ball in L_B^∞ , then there exists a neighbourhood $V_1(m_1, \varepsilon_1, K_1)$ in $\mathcal{D}_\Omega^{m_1}$, such that $U_\psi * (\alpha * \beta) \in Z(0, \rho)$ for $\alpha, \beta \in V_1(m_1, \varepsilon_1, K_1)$, $U_\psi \in H$ and a neighbourhood $V_2(m_2, \varepsilon_2, K_2) \subset \mathcal{D}_\Omega^{m_2}$, such that $T_h * (\alpha * \beta) \in Z(0, \rho)$ for $\alpha, \beta \in V_2(m_2, \varepsilon_2, K_2)$, $T_h \in \mathcal{Q}$. Let $K_0 = K_1 \cap K_2$, $\varepsilon_0 = \min(\varepsilon_1, \varepsilon_2)$ and $m = \max(m_1, m_2)$. We shall now use relation (VI, 6; 23) from Schwartz [1], II:

$$T_0 = \Delta^{2k} * (\gamma E * \gamma E * T) - 2\Delta^k * (\gamma E * \xi * T) + (\xi * \xi * T), \quad (4.2)$$

where E is a solution of the iterated Laplace equation; $\Delta^k E = \delta$; $\gamma, \xi \in \mathcal{D}_\Omega$, $\text{supp} \gamma$ and $\text{supp} \xi$ belonging to $K_0 = K_1 \cap K_2$. We have only to choose the number k large enough so

that $\gamma E \in \mathcal{D}_\Omega^m$. Now, we can take: $F_2 = \gamma E * \gamma E * T_0$; $F_1 = \gamma E * \xi * T_0$ and $F_0 = \xi * \xi * T_0$. All of these functions are of the form: $F_i = T_0 * \alpha_i * \beta_i$; $\alpha_i, \beta_i \in V(m, \varepsilon'_0, K_0)$, $\varepsilon'_0 > 0$.

We have to prove that F_i have the properties given in Theorem 1. For $\alpha_i, \beta_i \in V(m, \varepsilon'_0, K_0)$ and $\psi \in S$

$$| \langle (T_0 * \alpha_i * \beta_i), \psi \rangle | = | \langle [(T_0 * \hat{\psi}) * (\hat{\alpha}_i * \hat{\beta}_i)](0) \rangle | \leq \rho(\varepsilon'_0 / \varepsilon_0)^2 \equiv M \quad (4.3)$$

Now let $\mu \neq 0$ be any element from L^1 , then $\mu / \|\mu\|_{L^1} \in S$ and $| \langle (T_0 * (\alpha_i * \beta_i)), \mu \rangle | \leq M \|\mu\|_{L^1}$ which proves that $T_0 * (\alpha_i * \beta_i)$ belong to L^∞ . Since $F_i = T_0 * (\alpha_i * \beta_i)$, $\alpha_i, \beta_i \in V(m, \varepsilon'_0, K_0)$, F_i are continuous and belong to L^∞ .

c) We shall continue with investigations of the properties of F_i . By the properties of the convolution we have that $F_i(x+h) = F_i * \tau_{-h} = T_0 * (\alpha_i * \beta_i) * \tau_{-h} = T_h * (\alpha_i * \beta_i)$ for $\alpha_i, \beta_i \in \mathcal{D}_\Omega^m$, where τ_{-h} is the translation operator.

We have proved that the mappings $(\alpha, \beta) \rightarrow T_h * \alpha * \beta$, $T_h \in \mathcal{Q}$, are equicontinuous and map $\mathcal{D}_\Omega^m \times \mathcal{D}_\Omega^m$ into L_B^∞ . $\mathcal{D}$ is a dense subset of $\mathcal{D}^m$, $m \geq 0$. We can construct a subset of $\mathcal{D}_K$, $\text{cl } \mathcal{D} = K$, which is dense in $\mathcal{D}_\Omega^m$. Since $T_h * (\zeta * \psi) \rightarrow C * \zeta * \psi$ for $\zeta * \psi \in \mathcal{D}_\Omega \times \mathcal{D}_\Omega$, then $T_h * \alpha_i * \beta_i$ converges to $C * \alpha_i * \beta_i$, as well (Schwartz [1], II, p.53).

d) there remains to prove the last part of Theorem 1. For $\mu \in \mathcal{D}(L^1)$ and $T \in \mathcal{B}'$, noting that $F_i \in L^\infty$, we have:

$$| \langle T_0(x+h), \mu(x) \rangle | \leq \sum_{i=0}^2 \int_{\mathbb{R}^n} |F_i(x+h) \Delta^{ik} \mu(x)| dx \leq \sum_{i=0}^2 M_i \int_{\mathbb{R}^n} |\Delta^{ik} \mu(x)| dx, \quad (4.4)$$

where $M_i = \sup |F_i(x)|$, $x \in \mathbb{R}^n$. Hence the set $\{T_0(x+h), h \in \mathbb{R}^n\}$, is weakly bounded in $\mathcal{B}'$. Since $\mathcal{D}$ is dense in $\mathcal{D}(L^1)$, by the Banach-Steinhaus theorem the limit:

$$\lim_{h \in \Gamma, h \rightarrow \infty} \langle T_0(x+h), \mu(x) \rangle, \mu \in \mathcal{D}(L^1), \quad (4.5)$$

exists, as well, and equals $\langle C, \mu \rangle$.

5. ABELIAN THEOREMS FOR THE STIELTJES TRANSFORM

THEOREM 2. Suppose that $T \in \mathcal{D}'$, the cone Γ is with the nonempty interior and

- i) $T(x+h) \approx c(h) \cdot U(x)$, $h \in \Gamma$ in $\mathcal{D}'$;
- ii) For a $r > 0$ and $s_0 \in (C \setminus \mathbb{R})^n$ the distribution $T(x)/(s_0+x)^r$ belongs to $\mathcal{B}'$;
- iii) For the same r and s_0 , $c(h)/(s_0+x+h)^r$ converges to $C_1 \neq 0$ when $h \in \Gamma$, $h \rightarrow \infty$ and x belongs to any compact set in $\mathbb{R}^n$.

Then T has the S_ρ -transform for all $\rho > r$, $S_\rho(T)(s) = \langle T(x)/(s_0+x)^r, (s_0+x)^r/(s+x)^{\rho+e} \rangle$ and

$$\lim_{h \in \Gamma, h \rightarrow \infty} S_\rho(T)(s-x) c(h) = 0, \rho > r. \quad (5.1)$$

PROOF. From supposition iii) it follows that the limit distribution $U = C$. Namely, for a $y \in \mathbb{R}^n$, x belonging to a compact set in $\mathbb{R}^n$ and $s_0 \in (C \setminus \mathbb{R})^n$ from the relation:

$$\frac{c(h+y)}{c(h)} = \frac{c(h+y)}{(s_0+x+h+y)^r} \frac{(s_0+x+h)^r}{c(h)} \frac{(s_0+x+h+y)^r}{(s_0+x+h)^r} \quad (5.2)$$

and from the Lemma 1 with the Remark after Lemma 1, we have $\lim_{h \in \Gamma, h \rightarrow \infty} c(h+y)/c(h) = 1$. Now, relation (2.6) gives $U = C$.

By Theorem X from [1], I, p. 72 (see also Pilipović and Stanković [2]), for a $\zeta \in \mathcal{D}$ it follows the existence of the limit:

$$\lim_{h \in \Gamma, h \rightarrow \infty} \langle T(x+h)/(s_0+x+h)^r, \zeta(x) \rangle = \lim_{h \in \Gamma, h \rightarrow \infty} \langle \frac{T(x+h)}{c(h)}, \frac{c(h)}{(s_0+x+h)^r} \zeta(x) \rangle = \quad (5.3)$$

$$= \langle CC_1, \zeta(x) \rangle.$$

In such a way we proved that the distribution $T(x)/(s_0+x)^r$, $s_0 \in (C \setminus R)^n$, has the same properties as the distribution T_0 from Theorem 1 and we can use the assertion of this theorem.

We shall prove, now, the existence of the S_ρ -transform for $\rho > r$.

Since $|D_\omega^k \eta_\omega(x)| \leq C_k \omega^{(-k \cdot e)}$, $k \geq 0$, where C_k do not depend on ω , the set of functions $\eta_\omega(x)(s+x)^{r-\rho-e}$ converges to $(s+x)^{r-\rho-e}$, $\rho > r$, in $\mathcal{D}(L^1)$, when $\omega \rightarrow \infty$. Moreover, $(s_0+x)^r/(s+x)^r$, for $s_0, s \in (C \setminus R)^n$ belongs to $\mathcal{D}(L^\infty)$; consequently

$$S_\rho(T)(s) = \lim_{\omega \rightarrow \infty} \langle T(x), \eta_\omega(x)(s+x)^{-\rho-e} \rangle = \lim_{\omega \rightarrow \infty} \langle \frac{T(x)}{(s_0+x)^r} \frac{(s_0+x)^r}{(s+x)^r}, \eta_\omega(x)(s+x)^{r-\rho-e} \rangle = \quad (5.4)$$

$$= \langle T(x)/(s_0+x)^r, (s_0+x)^r(s+x)^{-\rho-e} \rangle, \quad \rho > r.$$

We have seen that the distribution $T(x)/(s_0+x)^r$ satisfies the conditions of Theorem 1, therefore

$$\frac{1}{c(h)} S_\rho(T)(s-h) = \frac{1}{c(h)} \langle \frac{T(x+h)}{(s_0+x+h)^r}, (s_0+x+h)^r(s+x)^{-\rho-e} \rangle = \quad (5.5)$$

$$= \int_{\mathbb{R}^n} \frac{(-1)^{i(k \cdot e)}}{c(h)} \int_{\mathbb{R}^n} F_i(x+h) \Delta^{ik} \frac{(s_0+x+h)^r}{(s+x)^{\rho+e}} dx.$$

The expression $\Delta^{ik}[(s_0+x+h)^r(s+x)^{-\rho-e}]$ is given by the finite sum of elements which have the following form:

$$H_{j,p} = C_{j,p} (s_0+x+h)^{r-j+p} (s+x)^{-\rho-e-p}, \quad j \geq p \geq 0. \quad (5.6)$$

We shall analyse $H_{j,p}/c(h)$ when $x \in \mathbb{R}^n$ and $h \in \Gamma$.

First we prove two inequalities:

$$|(s_0+x+h)^r| = \prod_{i=1}^n |s_{0,i}+x_i+h_i|^{r_i} \leq \prod_{i=1}^n (|s_{0,i}+h_i| + 1)^{r_i} (|x_i|+1)^{r_i} \leq \quad (5.7)$$

$$\leq \prod_{i=1}^n |s_{0,i}+h_i|^{r_i} \left(1 + \frac{1}{|\operatorname{Im} s_{0,i}|}\right)^{r_i} \prod_{i=1}^n (|x_i| + 1)^{r_i} \leq$$

$$\leq C_r |(s_0+h)^r| \prod_{i=1}^n (|x_i|+1)^{r_i};$$

$$|(s_0+x+h)^{p-j}| = \prod_{i=1}^n |s_{0,i}+x_i+h_i|^{p_i-j_i} \leq \prod_{i=1}^n |\operatorname{Im} s_{0,i}|^{p_i-j_i} = C'_{p,j}. \quad (5.8)$$

Now

$$|H_{j,p}|/c(h) \leq C_{j,p} C'_{j,p} C_r \left| \frac{(s_0+h)^r}{c(h)} \right| \prod_{i=1}^n \frac{(|x_i|+1)^{r_i}}{(s_i+x_i)^{\rho_i+p_i+1}}, \quad p \geq 0. \quad (5.9)$$

This inequality shows that $|H_{j,p}|/c(h)$ is bounded by a function which belongs to L^1 , when $h \in \Gamma$. Since $F_i(x+h)$ are bounded, as well, when $x \in \mathbb{R}^n$ and $h \in \Gamma$, we can use the Lebesgue theorem for the integral in (5.5) to obtain that $S_\rho(T)(s-h)/c(h)$ tends to zero when $h \in \Gamma$, $h \rightarrow \infty$.

The next theorem presents more precisely how $S_\rho(T)(s-h)$ tends to zero when $h \in \Gamma_2$, $h \rightarrow \infty$. Γ_2 is defined in the Remark after Lemma 1. We have seen that $|h_i| \rightarrow \infty$, $i = 1, \dots, n$, if $h \rightarrow \infty$, $h \in \Gamma_2$.

THEOREM 3. Let $T \in \mathcal{D}'$ and $s_0 \equiv (s_{0,1} \dots s_{0,n}) \in (C \setminus R)^n$. We suppose:

- i) $(s_{0,j} + x_j + h_j)T(x+h) \stackrel{s}{\sim} 1 \cdot C$, $h \in \Gamma$ in $\mathcal{D}'$; $1 \leq j \leq n$
- ii) $(s_{0,j} + x_j)T(x) \in B'$.

Then, for $\rho > 0$ and Γ convex cone, $\overset{\circ}{\Gamma} \neq \emptyset$

$$\lim_{h \in \Gamma_2, h \rightarrow \infty} h_\rho S_\rho(T)(s-h) = 0 \quad (5.10)$$

PROOF. We can write:

$$\begin{aligned} S_\rho((s_{0,j} + x_j)T(x))(s-h) &= \langle (s_{0,j} + x_j + h_j)T(x+h), (s+x)^{-\rho-e} \rangle = \\ &= \langle T(x+h), (s+x)^{-\rho-e'} \rangle + (s_{0,j} - s_j + h_j) \langle T(x+h)(s+x)^{-\rho-e} \rangle; \quad e' = (e'_1 \dots e'_n), \end{aligned} \quad (5.11)$$

where $e'_i = 1$, $i \neq j$ and $e'_j = 0$. Hence

$$S_\rho(T)(s-h) = (s_{0,j} - s_j + h_j)^{-1} [S_\rho((s_{0,j} + x_j)T(x))(s-h) - \langle T(x+h), (s+x)^{-\rho-e'} \rangle] \quad (5.12)$$

First, we shall prove that $S_\rho((s_{0,j} + x_j)T(x))(s-h) \rightarrow 0$ when $h \in \Gamma_2$, $h \rightarrow \infty$. By Theorem 1 it follows:

$$\lim_{h \in \Gamma_2, h \rightarrow \infty} S_\rho((s_{0,j} + x_j)T(x))(s-h) = \langle C, (s+x)^{-\rho-e} \rangle = 0 \quad (5.13)$$

It remains to prove that $\langle T(x+h), (s+x)^{-\rho-e'} \rangle \rightarrow 0$ when $h \in \Gamma_2$, $h \rightarrow \infty$, as well. By Theorem 1 we know that our distribution T has the form

$$T(x) = (s_{0,j} + x_j)^{-1} \sum_{i=0}^2 \Delta^{ik} F_i(x) \quad (5.14)$$

where $F_i(x)$ are bounded functions when $x \in \mathbb{R}^n$. Using this form of T , we have:

$$\langle T(x+h), (s+x)^{-\rho-e'} \rangle = \sum_{i=0}^2 (-1)^{i(k \cdot e)} \langle F_i(x+h), \Delta^{ik} (s_{0,j} + x_j + h_j)^{-1} (s+x)^{-\rho-e'} \rangle \quad (5.15)$$

The second part of relation (5.15) consists in fact of a sum of integrals

$$\int_{\mathbb{R}^n} F_i(x+h) (s_{0,j} + x_j + h_j)^{-m} (s+x)^{-\rho-\alpha} dx \quad (5.16)$$

where $m \geq 1$; $\alpha_j \geq 0$ and $\alpha_i \geq 1$, $i \neq j$. All of these integrals tend to zero when $h \in \Gamma_2$, $h \rightarrow \infty$. We shall prove only the case: $m = 1$, $\alpha_j = 0$. In other cases it is trivial.

Let us consider the integral

$$\int_{-\infty}^{\infty} F_i(x+h) (s_{0,j} + x_j + h_j)^{-1} (s_j + x_j)^{-\rho_j} dx_j \quad (5.17)$$

for $x_k \in \mathbb{R}$, $k \neq j$ and $h \in \Gamma_2$. The function in x_j $F_i(x+h) (s_{0,j} + x_j + h_j)^{-1}$ belongs to $\mathcal{D}'(L^V)$ for every $v > 1$. Let p be such that $p\rho_j > 1$ and $\frac{1}{p} - \frac{1}{v} - 1 \geq 0$. Then, for $x_i \in \mathbb{R}$, $i \neq j$, integral (5.17) is bounded by a function in h belonging to $\mathcal{D}(L^u)$, $\frac{1}{u} = \frac{1}{v} + \frac{1}{p} - 1$ (Schwartz [1], II, p. 60).

Now, we have to prove that we can find p , satisfying our conditions, and $v > 1$, such that the number u remains in the interval $1 \leq u < \infty$. Then, we have only to use the

property of a $\zeta \in \mathcal{D}(L^u)$, $1 \leq u < \infty$; namely: $\zeta(x) \rightarrow 0$, $\|x\| \rightarrow \infty$ (Schwartz [1], II, p. 55) and it will follow that integral (5.16) tend to zero when $h \in \Gamma_2$, $h \rightarrow \infty$. Therefore we shall analyse two cases.

In case $\rho_j \geq 1$, we can take $\frac{1}{v} = 1 - \frac{1}{2v}$, hence $u = 2v$.

Case $0 < \rho_j < 1$. Since $\frac{1}{v} + \frac{1}{p} - 1 \geq 0$, we have $1 > \frac{1}{v} \geq 1 - \frac{1}{p} > 1 - \rho_j$, or $1 < v < (1 - \rho_j)^{-1}$. For such an v the number $1/u$ is strictly positive, hence $1 \leq u < \infty$.

The next example shows that Theorem 3 cannot be proved for $\rho = 0$

$$\int_0^\infty \frac{dx}{(x+a)(s+x)} = (a-s)^{-1} \ln\left(\frac{a}{s}\right), \quad a, s > 0. \quad (5.18)$$

There arises another question: If we know that $(s_{0,j} + x_j)^r T(x) \in B'$ for a $r > 1$, is it true that $S_\rho(T)(s-h) = o(h_j^{-r})$, $h \in \Gamma_2$, $h \rightarrow \infty$? The answer is in general negative. This shows the following integral:

$$\int_0^\infty e^{-t} \frac{dt}{(s+t)^v} = s^{-v/2} \cdot \exp(s/2) W_{-v/2, (1-v)/2}(s) \sim s^{-v}, \quad s > 0, s \rightarrow \infty, \quad (5.19)$$

where $W_{\nu, \mu}$ is the Whittaker function.

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REFERENCES

1. SCHWARTZ, L. Théorie des distributions, T. I-II, Hermann, Paris, 1950-1951.
2. PILIPOVIĆ, S. and STANKOVIĆ, B. S-asymptotic of a distribution, Pliska Stud. Math. Bulgar., Vol 10, (1989), 147-156.
3. STANKOVIĆ, B. S-asymptotic expansion of distributions, Internat. J. Math. Math. Sci. Vol. 11, No. 3 (1988), 449-456.
4. VLADIMIROV, V.S. Generalized Functions in Mathematical Physics, MIR Publisher, Moscow, 1979.
5. LAVOINE, J. and MISRA, O.P. Théorèmes Abéliens pour la transformation de Stieltjes des distributions, C. R. Acad. Sc. Paris Sér. I Math. T 279 (1974), 99-102.
6. LAVOINE, J. and MISRA, O.P. Sur la transformation de Stieltjes des distributions et son inversion au moyen de la transformation de Laplace, C. R. Acad. Sc. Paris Sér. I Math. T. 290 (1980), 139-142.
7. ZEMANIAN, A.H. Generalized integral transformations, Intersciences, New York, 1986.
8. PANDEY, J.N. On the Stieltjes transform of generalized functions, Math. Proc. Cambridge Philos. Soc., 71 (1971), 85-96.
9. BREMERMAN, H. Distributions, Complex Variables and Fourier Transforms, Addison-Wesley, Reading, Mass, 1965.
10. PILIPOVIĆ, S. and STANKOVIĆ, B. Abelian theorem for the distributional Stieltjes transform, Z. Anal. Anwendungen Bd. 6(4), (1978), 341-349.