ORTHODOX Γ -SEMIGROUPS

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ABSTRACT. Let $M = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two non-empty sets. M is called a Γ -semigroup if $a\alpha b \in M$, for $\alpha \in \Gamma$ and $b \in M$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$. A semigroup can be considered as a Γ -semigroup. In this paper we introduce orthodox Γ -semigroups and extend different results of orthodox semigroups to orthodox Γ -semigroups.

KEY WORDS AND PHRASES. Semigroup, Γ -semigroup, Orthodox Γ -semigroup, Inverse Γ -semigroup.

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1. INTRODUCTION.

Let A and B be two non-empty sets, M the set of all mappings from A to B , and Γ a set of some mappings from B to A . The usual mapping product of two elements of M can not be defined. But if we take f, g from M and α from Γ then the usual mapping product $f\alpha g$ can be defined. Also we find that $f\alpha g \in M$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$ for $f, g, h \in M$ and $\alpha, \beta \in \Gamma$.

If M is the set of $m \times n$ matrices and Γ is a set of some $n \times m$ matrices over the field of real numbers, then we can define $A_{m,n} \alpha_{n,m} B_{m,n}$ such that

$$(A_{m,n} \alpha_{n,m} B_{m,n}) \beta_{n,m} C_{m,n} = A_{m,n} \alpha_{n,m} (B_{m,n} \beta_{n,m} C_{m,n}) \text{ where } A_{m,n}, B_{m,n}, C_{m,n} \in M \text{ and } \alpha_{n,m}, \beta_{n,m} \in \Gamma.$$

An algebraic system satisfying the associative property of the above type is a Γ -semigroup (Saha [1]).

DEFINITION 1.1. Let $M = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \dots\}$ be two non-empty sets. M is called a Γ -semigroup if (i) $a\alpha b \in M$ for $\alpha \in \Gamma$ and $a, b \in M$ and (ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in M$ and for all $\alpha, \beta \in \Gamma$.

A semigroup can be considered a Γ -semigroup in the following sense. Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S . Let us extend the given binary relation in S to $S \cup 1$ by defining $11 = 1$ and $1a = a1 = a$ for all a in S . It can be shown that $S \cup 1$ is a semigroup with identity element 1 .

Let $\Gamma = \{1\}$. Putting $ab = alb$ it can be shown that the semigroup S is a Γ -semigroup where $\Gamma = \{1\}$. Since every semigroup is a Γ -semigroup in the above sense, the main thrust of our work is to extend different fundamental results of semigroups to Γ -semigroups. In Sen and Saha [2] and Saha [1,3,4] we have extended some results of semigroups to Γ -semigroups. In this paper we want to introduce orthodox Γ -semigroups and we want to extend results of Hall [5] and Yamada [6] to Γ -semigroups.

2. PRELIMINARIES.

We recall the following definitions and results from [1], [2], [3] and [4].

DEFINITION 2.1. Let M be a Γ -semigroup. A non-empty subset B of M is said to be a Γ -subsemigroup of M if $B\Gamma B \subset B$.

DEFINITION 2.2. Let M be a Γ -semigroup. An element $a \in M$ is said to be regular if $a \in a\Gamma a$, where $a\Gamma a = \{a\alpha\beta a : b \in M, \alpha, \beta \in \Gamma\}$. A Γ -semigroup M is said to be regular if every element of M is regular.

EXAMPLE 2.1. Let M be the set of 3×2 matrices and Γ be a set of some 2×3 matrices over a field. We show that M is a regular Γ -semigroup. Let $A \in M$, where $A = \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$. Then we choose $B \in \Gamma$ according to the following cases such that $ABABA = ABA = A$.

CASE 1. When the submatrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular, then $ad - bc \neq 0$. e, f may both be 0 or one of them is 0 or both of them are non-zero.

Then $B = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} & 0 \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} & 0 \end{pmatrix}$ and we find $ABA = A$.

CASE 2. $af - be \neq 0$. Then $B = \begin{pmatrix} \frac{f}{af-be} & 0 & \frac{-b}{af-be} \\ \frac{-e}{af-be} & 0 & \frac{a}{af-be} \end{pmatrix}$ and $ABA = A$.

CASE 3. $cf - de \neq 0$. Then $B = \begin{pmatrix} 0 & \frac{f}{cf-de} & \frac{-d}{cf-de} \\ 0 & \frac{-e}{cf-de} & \frac{c}{cf-de} \end{pmatrix}$ and $ABA = A$.

CASE 4. When the submatrices are singular. Then either $\begin{cases} ad - bc = 0 \text{ or} \\ ad - bc = 0, \\ af - de = 0 \end{cases}$ $\begin{cases} cf - be = 0 \end{cases}$

If all the elements of A are 0, then the case is trivial. Next we consider at least one of the elements of A is non-zero, say $a_{ij} \neq 0$, $i = 1, 2, 3$ and $j = 1, 2$. Then we take the b_{ji} th element of B as $(a_{ij})^{-1}$ and the other elements of B are zero and we find that $ABA = A$. Thus A is regular. Hence M is a regular Γ -semigroup.

DEFINITION 2.3. Let M be a Γ -semigroup. An element $e \in M$ is said to be an idempotent of M if there exists an $\alpha \in \Gamma$ such that $e\alpha e = e$. In this case we shall say e is an α -idempotent.

DEFINITION 2.4. Let M be a Γ -semigroup and $a \in M$. Let $b \in M$ and $\alpha, \beta \in \Gamma$. b is said to be an (α, β) inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. In this case we shall write $b \in V_{\alpha}^{\beta}(a)$.

DEFINITION 2.5. A regular Γ -semigroup M is called an inverse Γ -semigroup if $|V_{\alpha}^{\beta}(a)| = 1$, for all $a \in M$ and for all $\alpha, \beta \in \Gamma$, whenever $V_{\alpha}^{\beta}(a) \neq \emptyset$. That is, every element a of M has a unique (α, β) inverse whenever the (α, β) inverse of a exists.

THEOREM 2.1. Let M be a Γ -semigroup. M is an inverse Γ -semigroup if and only if (i) M is regular and (ii) if e and f are any two α -idempotents of M then $e\alpha f = f\alpha e$, where $\alpha \in \Gamma$.

LEMMA 2.2. Let M be a regular Γ -semigroup and let M' be a Γ' -semigroup. Let (f, g) be a homomorphism from (M, Γ) onto (M', Γ') . Then M' is a regular Γ' -semigroup.

3. ORTHODOX Γ -SEMIGROUP.

DEFINITION 3.1. A regular Γ -semigroup M is called an orthodox Γ -semigroup if for e an α -idempotent and f a β -idempotent then $e\alpha f$, $f\alpha e$ are β -idempotents and $e\beta f$, $f\beta e$ are α -idempotents.

EXAMPLE 3.1. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. M denotes the set of all mappings from A to B . Here members of M will describe the images of the elements 1, 2, 3. For example the map $1 \mapsto 4$, $2 \mapsto 5$, $3 \mapsto 4$ will be written as $(4, 5, 4)$ and $(4, 4, 5)$ denotes the map $1 \mapsto 4$, $2 \mapsto 4$, $3 \mapsto 5$. Again a map from $B \rightarrow A$ will be in the same fashion. For example $(1, 2)$ denotes $4 \mapsto 1$, $5 \mapsto 2$. Now, $M = \{(4, 4, 4), (4, 4, 5), (4, 5, 4), (4, 5, 5), (5, 5, 5), (5, 4, 5), (5, 4, 4), (5, 5, 4)\}$. Let $\Gamma = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 3)\}$ be a set of some mappings from B to A . Let $f, g \in M$ and $\alpha \in \Gamma$. Under the usual mapping composition, $f\alpha g$ is a mapping from A to B and hence $f\alpha g \in M$. Also, we can easily show that $(f\alpha g)\beta h = f\alpha(g\beta h)$, for all $f, g, h \in M$ and $\alpha, \beta \in \Gamma$. One can easily verify that M is a regular Γ -semigroup. Here

$$\begin{aligned} (4, 4, 4) (1, 1) (4, 4, 4) (1, 1) (4, 4, 4) &= (4, 4, 4) \\ (4, 4, 5) (1, 3) (4, 4, 5) (1, 3) (4, 4, 5) &= (4, 4, 5) \\ (4, 5, 4) (1, 2) (4, 5, 4) (1, 2) (4, 5, 4) &= (4, 5, 4) \\ (4, 5, 5) (1, 2) (4, 5, 5) (1, 2) (4, 5, 5) &= (4, 5, 5) \\ (5, 5, 5) (1, 1) (5, 5, 5) (1, 1) (5, 5, 5) &= (5, 5, 5) \\ (5, 4, 5) (1, 2) (5, 4, 4) (1, 2) (5, 4, 5) &= (5, 4, 5) \\ (5, 4, 4) (1, 2) (5, 4, 5) (1, 2) (5, 4, 4) &= (5, 4, 4) \end{aligned}$$

$$(5,5,4) (1,2) (5,4,5) (1,3) (5,5,4) = (5,5,4)$$

Here $(4,4,5)$ is $(1,3)$ idempotent, and $(5,4,4)$ is $(2,1)$ idempotent, but $(4,4,5) (1,3) (5,4,4) = (5,5,4)$ is not an idempotent. Hence this regular Γ -semigroup is not an orthodox Γ -semigroup.

EXAMPLE 3.2. Let Q^* denote the set of all non-zero rational numbers. Let Γ be the set of all positive integers. Let $a \in Q^*$, $\alpha \in \Gamma$ and $b \in Q^*$. $a \circ b$ is defined by $|a|^\alpha |b|$. For this operation Q^* is a Γ -semigroup. Let $\frac{p}{q} \in Q^*$. Now $|\frac{p}{q}|^\alpha |\frac{1}{q}| = \frac{p}{q}$. Hence this is a regular Γ -semigroup. Here $\frac{1}{q}, |q| \in \Gamma$ is a $|q|$ -idempotent. These are the only idempotents of Q^* . Now $\frac{1}{q} |q|^\alpha \frac{1}{p}$ is a $|p|$ -idempotent. Hence Q^* is an orthodox Γ -semigroup.

THEOREM 3.3. Every inverse Γ -semigroup is an orthodox Γ -semigroup.

PROOF. Let M be an inverse Γ -semigroup. Let e be a α -idempotent and f be a β -idempotent. Now $e \circ f \in M$. Since M is an inverse Γ -semigroup, let $x \in V_\delta^\gamma(e \circ f)$. Then $e \circ f \circ x \circ y \circ e \circ f = e \circ f$, $x \circ y \circ e \circ f \circ x = x$. Let $g = f \circ x \circ y \circ e \circ f$. Then $g \circ g = g$. Also, let $h = f \circ x \circ y \circ e$. Then, $f \circ x \circ y \circ e \circ f \circ x \circ y \circ e \circ f \circ x \circ y \circ e \circ f = f \circ x \circ y \circ e \circ f \circ x \circ y \circ e \circ f = f \circ x \circ y \circ e \circ f = g$. This shows that $g \circ h \circ g = g$. Similarly $h \circ g \circ h = h$. Hence $g \in V_\alpha^\beta(h)$. Also $e \circ f \in V_\alpha^\beta(h)$. Since M is an inverse Γ -semigroup, $g = e \circ f$. Hence $e \circ f$ is a β -idempotent. Similarly we can show that $f \circ e$ is β -idempotent, and both $e \circ f$ and $f \circ e$ are α -idempotents.

EXAMPLE 3.4. In example 3.2 we have shown that Q^* is an orthodox Γ -semigroup. Now $(1/q) \in V_p^1(q/p)$. Also $(-1/q) \in V_p^1(q/p)$. Hence Q^* is not an inverse Γ -semigroup.

THEOREM 3.5. A regular Γ -semigroup M is an orthodox Γ -semigroup if and only if for any α -idempotent $e \in M$, where $\alpha \in \Gamma$, if $V_\alpha^\beta(e) \neq \emptyset$ and $V_\beta^\alpha(e) \neq \emptyset$, then each member of $V_\alpha^\beta(e)$ and $V_\beta^\alpha(e)$ is a β -idempotent.

PROOF. Suppose M is an orthodox Γ -semigroup. Let e be an α -idempotent of M and let $x \in V_\alpha^\beta(e)$. Then $e \circ x \circ \beta e = e$ and $x \circ \beta e \circ \alpha = x$. Now $e \circ x$ is a β -idempotent and $x \circ \beta e$ is an α -idempotent. Then $x = (x \circ \beta e) \circ \alpha (e \circ x)$ is a β -idempotent. Next let $y \in V_\beta^\alpha(e)$. Then $e \circ \beta y \circ \alpha = e$ and $y \circ \alpha \circ \beta y = y$. Now $y \circ \alpha$ is a β -idempotent and $e \circ \beta y$ is an α -idempotent. Then $y = (y \circ \alpha) \circ \alpha (e \circ \beta y)$ is a β -idempotent. Conversely suppose that M satisfies the given conditions. Let e be an α -idempotent and f be a β -idempotent. Consider $e \circ f$. Now $e \circ f \in M$, and since M is regular there exists $x \in M$ and $\gamma, \delta \in \Gamma$ such that $e \circ f \circ x \circ \gamma \circ e \circ f = e \circ f$ and $x \circ \delta \circ e \circ f \circ x = x$. Let $g = f \circ x \circ \delta \circ e$. Then $g \circ g = f \circ x \circ \delta \circ e \circ f \circ x \circ \delta \circ e \circ f = f \circ x \circ \delta \circ e = g$. Now, $e \circ f \circ \beta f \circ x \circ \delta \circ e \circ \alpha \circ f = e \circ f \circ x \circ \delta \circ e \circ f = e \circ f$ and $f \circ x \circ \delta \circ e \circ \alpha \circ f \circ \beta f \circ x \circ \delta \circ e = f \circ x \circ \delta \circ e \circ f \circ x \circ \delta \circ e = f \circ x \circ \delta \circ e$. Hence $e \circ f \in V_\alpha^\beta(f \circ x \circ \delta \circ e)$. Then by the given condition $e \circ f$ is β -idempotent. Dually we can prove that $f \circ e$ is β -idempotent. Similarly, it is easy to see that $e \circ f$ and $f \circ e$ are α -idempotents.

THEOREM 3.6. A regular Γ -semigroup M is an orthodox Γ -semigroup if and only if

for $a, b \in M$, $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Gamma$, $a' \in V_{\alpha_1}^{\alpha_2}(a)$, and $b' \in V_{\beta_1}^{\beta_2}(b)$ we have

$$b' \beta_2 a' \in V_{\beta_1}^{\alpha_2}(a \alpha_1 b) \text{ and } b' \alpha_1 a' \in V_{\beta_1}^{\alpha_2}(a \beta_2 b).$$

PROOF. Let us assume that M is an orthodox Γ -semigroup. Let $a' \in V_{\alpha_1}^{\alpha_2}(a)$ and $b' \in V_{\beta_1}^{\beta_2}(b)$. Then $a\alpha_1 a' \alpha_2 a = a$, $a' \alpha_2 a \alpha_1 a' = a'$, $b\beta_1 b' \beta_2 b = b$, and $b' \beta_2 b \beta_1 b' = b'$. Now $a' \alpha_2 a$ is an α_1 -idempotent and $b\beta_1 b'$ is a β_2 -idempotent. Hence $(a' \alpha_2 a) \alpha_1 (b\beta_1 b')$ is a β_2 -idempotent, and $(b\beta_1 b') \beta_2 (a' \alpha_2 a)$ is an α_1 -idempotent. $(a' \alpha_2 a) \beta_2 (b\beta_1 b')$ is an α_1 -idempotent and $(b\beta_1 b') \alpha_1 (a' \alpha_2 a)$ is a β_2 -idempotent.

$$\begin{aligned} a\alpha_1 b\beta_1 b' \beta_2 a' \alpha_2 a \alpha_1 b &= a\alpha_1 a' \alpha_2 a \alpha_1 b\beta_1 b' \beta_2 a' \alpha_2 a \alpha_1 b\beta_1 b' \beta_2 b \\ &= a\alpha_1 a' \alpha_2 a \alpha_1 b\beta_1 b' \beta_2 b \quad (\text{since } a' \alpha_2 a \alpha_1 b\beta_1 b' \text{ is } \beta_2\text{-idempotent}) \\ &= a\alpha_1 b. \end{aligned}$$

$$\begin{aligned} b' \beta_2 a' \alpha_2 a \alpha_1 b\beta_1 b' \beta_2 a' &= b' \beta_2 b\beta_1 b' \beta_2 a' \alpha_2 a \alpha_1 b\beta_1 b' \beta_2 a' \alpha_2 a \alpha_1 a' \\ &= b' \beta_2 b\beta_1 b' \beta_2 a' \alpha_2 a \alpha_1 a' \quad (\text{since } b\beta_1 b' \beta_2 a' \alpha_2 a \text{ is } \alpha_1\text{-idempotent}) \\ &= b' \beta_2 a'. \end{aligned}$$

Hence $b' \beta_2 a' \in V_{\beta_1}^{\alpha_2}(a\alpha_1 b)$. Similarly it can be shown that $b' \alpha_1 a' \in V_{\beta_1}^{\alpha_2}(a\beta_2 b)$.

Conversely, assume that the given conditions hold in M . Let e be an α -idempotent and f be a β -idempotent of M . Now $f \in V_{\beta}^{\beta}(f)$ and $e \in V_{\alpha}^{\alpha}(e)$. Then by the given conditions (i) $e\alpha f \in V_{\alpha}^{\beta}(f\beta e)$ and (ii) $e\beta f \in V_{\alpha}^{\beta}(f\alpha e)$. From (i) we get $e\alpha f\beta f\beta e\alpha e\alpha f = e\alpha f$.

Then $e\alpha f\beta e\alpha f = e\alpha f$. Thus $e\alpha f$ is a β -idempotent. From (ii) we get $f\alpha e\alpha e\beta f\beta f\alpha e = f\alpha e$.

Then $f\alpha e\beta f\alpha e = f\alpha e$. So, $f\alpha e$ is β -idempotent. Again

$e \in V_{\alpha}^{\alpha}(e)$ and $f \in V_{\beta}^{\beta}(f)$. Then by the given conditions we get (iii) $f\beta e \in V_{\beta}^{\alpha}(e\alpha f)$ and (iv) $f\alpha e \in V_{\beta}^{\alpha}(e\beta f)$. From (iii) we get $f\beta e\alpha e\alpha f\beta f\beta e = f\beta e$. Then $f\beta e\alpha f\beta e = f\beta e$.

Hence $f\beta e$ is α -idempotent. From (iv) we get $e\beta f\beta f\alpha e\alpha e\beta f = e\beta f$. So, $e\beta f\alpha e\beta f = e\beta f$.

Thus $e\beta f$ is α -idempotent. Hence M is an orthodox Γ -semigroup.

THEOREM 3.7. A regular Γ -semigroup M is an orthodox Γ -semigroup if and only if for $a, b \in M$, $V_{\alpha}^{\beta}(a) \cap V_{\alpha}^{\beta}(b) \neq \emptyset$ for some $\alpha, \beta \in \Gamma$. This implies that

$$V_{\gamma}^{\delta}(a) = V_{\gamma}^{\delta}(b) \text{ for all } \gamma, \delta \in \Gamma.$$

PROOF. Suppose M is an orthodox Γ -semigroup. For $a, b \in M$, let there exist

$\alpha, \beta \in \Gamma$ such that $V_{\alpha}^{\beta}(a) \cap V_{\alpha}^{\beta}(b) \neq \emptyset$. Let $\gamma, \delta \in \Gamma$. First let us show that

$V_{\gamma}^{\delta}(a) \subset V_{\gamma}^{\delta}(b)$. Let $a' \in V_{\alpha}^{\beta}(a) \cap V_{\alpha}^{\beta}(b)$, and $a^* \in V_{\gamma}^{\delta}(a)$. Then $a\alpha a' \beta a = a$,

$a' \beta a \alpha a' = a'$, $b\alpha a' \beta b = b$, $a' \beta b \alpha a' = a'$, $a\gamma a^* \delta a = a$, $a^* \delta a \gamma a^* = a^*$. We can easily show that

$$(a^* \delta a) \alpha(a' \beta b) \gamma(a^* \delta a). \quad (3.1)$$

Now $a^* \delta a$ is γ -idempotent, and $a' \beta b$ is α -idempotent. Hence $(a^* \delta a) \alpha(a' \beta b)$ is γ -idempotent. Then from (3.1) we get

$$(a^* \delta a) \alpha(a' \beta b) \gamma(a^* \delta a) = a^* \delta a. \quad (3.2)$$

Now $a' \beta a = a' \beta a \gamma a^* \delta a = a' \beta a \gamma a^* \delta a \alpha a' \beta b \gamma a^* \delta a = a' \beta a \alpha a' \beta b \gamma a^* \delta a = a' \beta b \gamma a^* \delta a$.

$$\text{Hence } b \alpha a' \beta a = b \alpha a' \beta b \gamma a^* \delta a = b \gamma a^* \delta a. \quad (3.3)$$

Again we can show that $(b \alpha a') \beta(a \gamma a^*) \gamma(a \gamma a^*)$. Now $(b \alpha a') \beta(a \gamma a^*)$ is δ -idempotent.

Hence $(a \gamma a^*) \delta(b \alpha a') \beta(a \gamma a^*) = a \gamma a^*$.

Then $a \alpha a' = a \gamma a^* \delta a \alpha a' = a \gamma a^* \delta b \alpha a' \beta a \gamma a^* \delta a \alpha a' = a \gamma a^* \delta b \alpha a' \beta a \alpha a' = a \gamma a^* \delta b \alpha a'$

$$a \alpha a' \beta b = a \gamma a^* \delta b \alpha a' \beta b = a \gamma a^* \delta b. \quad (3.4)$$

$$\begin{aligned} \text{Now, } b \gamma a^* \delta b &= b \gamma a^* \delta a \gamma a^* \delta b \\ &= b \gamma a^* \delta a \alpha a' \beta b \quad \text{by (3.4)} \\ &= b \alpha a' \beta a \alpha a' \beta b \quad \text{by (3.3)} \\ &= b \alpha a' \beta b = b \end{aligned}$$

$$\begin{aligned} \text{and } a^* \delta b \gamma a^* &= a^* \delta a \gamma a^* \delta b \gamma a^* \delta a \gamma a^* \\ &= a^* \delta a \gamma a^* \delta b \alpha a' \beta a \gamma a^* \quad \text{by (3.3)} \\ &= a^* \delta a \alpha a' \beta b \alpha a' \beta a \gamma a^* \quad \text{by (3.4)} \\ &= a^* \delta a \alpha a' \beta a \gamma a^* = a^* \delta a \gamma a^* = a^*. \end{aligned}$$

Hence $a^* \in V_Y^\delta(b)$. Thus $V_Y^\delta(a) \subset V_Y^\delta(b)$. Similarly $V_Y^\delta(b) \subset V_Y^\delta(a)$. Thus $V_Y^\delta(a) = V_Y^\delta(b)$.

Conversely, assume that the given condition holds in M . Let e be α -idempotent and f be β -idempotent. Consider eaf . Since M is regular, there exists $\gamma, \delta \in \Gamma$ and $x \in M$ such that $eaf\gamma x\delta eaf = eaf$ and $x\delta eaf\gamma x = x$. Let $g = f\gamma x\delta e$. Then $g\alpha g = g$. Hence $f\gamma x\delta e \in V_\alpha^\alpha(f\gamma x\delta e)$. Let $h = eaf\gamma x\delta e$. Then $h\alpha h = h$. Also, $f\gamma x\delta e \in V_\alpha^\alpha(eaf\gamma x\delta e)$. Hence $V_\alpha^\alpha(g) \cap V_\alpha^\alpha(h) \neq \emptyset$. Then $V_\alpha^\theta(g) = V_\alpha^\theta(h)$ for any $\theta \in \Gamma$. But $eaf\beta f\gamma x\delta e\alpha eaf = eaf$ and $f\gamma x\delta e\alpha eaf\beta f\gamma x\delta e = f\gamma x\delta e$. Hence $eaf \in V_\alpha^\beta(f\gamma x\delta e)$. Then $eaf \in V_\alpha^\beta(eaf\gamma x\delta e)$. This implies that $eaf\beta eaf\gamma x\delta e\alpha eaf = eaf$. So $eaf\beta eaf = eaf$. Hence eaf is β -idempotent. Similarly, it can be proved that $f\alpha e$ is a β -idempotent and both $e\beta f$ and $f\beta e$ are α -idempotents. Let M be a regular Γ -semigroup and $a, b \in M$, $a' \in V_\alpha^\beta(a)$, and $b' \in V_Y^\delta(b)$. Then $e = a' \beta a$ is α -idempotent and $f = b \gamma b'$ is δ -idempotent. Let $\theta \in \Gamma$. Suppose $x \in V_{\alpha_1}^{\beta_1}(e\theta f)$. Then $e\theta f = e\theta f\alpha_1 x\beta_1 e\theta f$ and $x = x\beta_1 e\theta f\alpha_1 x$. Let $g = f\alpha_1 x\beta_1 e$. Now $g\theta g = f\alpha_1 x\beta_1 e\theta f\alpha_1 x\beta_1 e = f\alpha_1 x\beta_1 e = g$. Hence g is θ -idempotent.

Also $g\alpha e = g = f\delta g$ and $e\theta g\theta f = e\theta f$. Now,

$$\begin{aligned} (a\theta b)\gamma(b'\delta g\alpha a')\beta(a\theta b) &= a\theta(b\gamma b')\delta g\alpha(a'\beta a)\theta b = a\theta f\delta g\alpha e\theta b \\ &= a\theta f\delta g\theta b = a\theta g\theta b = a\alpha a'\beta a\theta g\theta b\gamma b'\delta b \\ &= a\alpha e\theta g\theta f\delta b = a\alpha e\theta f\delta b = a\theta b. \end{aligned}$$

Similarly we can show that $(\beta'\delta g\alpha a')\beta(a\theta b)\gamma(b'\delta g\alpha a') = b'\delta g\alpha a'$. Hence

$b'\delta g\alpha a' \in V_Y^\beta(a\theta b)$. Thus we have the following lemma.

LEMMA 3.8. Let M be a regular Γ -semigroup and $a, b \in M$. If $a' \in V_\alpha^\beta(a)$, $b' \in V_Y^\delta(b)$ and $\theta \in \Gamma$, then there exists a θ -idempotent $g \in M$ and $b'\delta g\alpha a' \in V_Y^\beta(a\theta b)$.

4. INVERSE Γ -SEMIGROUP CONGRUENCE.

DEFINITION 4.1. Let M be a Γ -semigroup. A congruence on M is defined as an equivalence relation ρ on the set M stable under left and right Γ -operations. That is, for every $a, b, c \in M$, $(a, b) \in \rho$ implies $(c\alpha a, c\alpha b) \in \rho$ and $(a\alpha c, b\alpha c) \in \rho$ for all $\alpha \in \Gamma$. A left (right) congruence on M is an equivalence relation on M , stable under left (right) Γ -operation.

Let M be a Γ -semigroup. Let ρ be a congruence on M . We define $(a\rho)\alpha(b\rho) = (a\alpha b)\rho$ for all $a\rho, b\rho \in M/\rho$ and for all $\alpha \in \Gamma$. It can easily be seen that the definition is well defined and M/ρ is a Γ -semigroup. Let us now characterize the minimum inverse Γ -semigroup congruence on an orthodox Γ -semigroup.

DEFINITION 4.2. Let M be an orthodox Γ -semigroup. A congruence ρ on M will be called an inverse Γ -semigroup congruence if M/ρ is an inverse Γ -semigroup.

THEOREM 4.1. Let M be an orthodox Γ -semigroup. Then the relation ρ defined by $\rho = \{(a, b) \in M \times M : V_\alpha^\beta(a) = V_\alpha^\beta(b) \text{ for all } \alpha, \beta \in \Gamma\}$ is the minimum inverse Γ -semigroup congruence on M .

PROOF. From the definition of ρ it is clear that ρ is an equivalence relation. To prove that ρ is a congruence relation, assume that $(a, b) \in \rho$, $c \in M$ and $\theta \in \Gamma$. Then $V_\alpha^\beta(a) = V_\alpha^\beta(b)$ for all $\alpha, \beta \in \Gamma$. Hence there exists $\alpha, \beta \in \Gamma$ such that $V_\alpha^\beta(a) = V_\alpha^\beta(b) \neq \phi$. Let $a^* \in V_\alpha^\beta(a) = V_\alpha^\beta(b)$, $c^* \in V_Y^\delta(c)$. Then by Lemma 3.8 $c^*\delta g\alpha a^* \in V_Y^\beta(a\theta c)$ and $c^*\delta g\alpha a^* \in V_Y^\beta(b\theta c)$, for some θ -idempotent $g = g\theta g \in M$. Hence $V_Y^\beta(a\theta c) \cap V_Y^\beta(b\theta c) \neq \phi$. Then by Theorem 3.7, $V_\alpha^\beta(a\theta c) = V_\alpha^\beta(b\theta c)$ for all $\alpha, \beta \in \Gamma$, so that

$(a\theta c, b\theta c) \in \rho$. Similarly we can show that $(c\theta a, c\theta b) \in \rho$. Hence ρ is a congruence on M . Suppose now $e = e\alpha e$ and $f = f\alpha f$ are two idempotents of M . Then $e\alpha f$ and $f\alpha e$ are α -idempotents and $e\alpha f \in V_{\alpha}^{\alpha}(e\alpha f)$ and $e\alpha f \in V_{\alpha}^{\alpha}(f\alpha e)$. Hence $V_{\alpha}^{\alpha}(e\alpha f) \cap V_{\alpha}^{\alpha}(f\alpha e) \neq \emptyset$. Consequently $V_Y^{\delta}(e\alpha f) = V_Y^{\delta}(f\alpha e)$ for all $\gamma, \delta \in \Gamma$. Thus we find that $(e\alpha f, f\alpha e) \in \rho$. Hence from Theorem 2.1 and Lemma 2.2 we find that M/ρ is an inverse Γ -semigroup. Finally, suppose that ρ_1 is a congruence on M such that M/ρ_1 is an inverse Γ -semigroup. If $(a, b) \in \rho$ then $V_{\alpha}^{\beta}(a) = V_{\alpha}^{\beta}(b)$ for all $\alpha, \beta \in \Gamma$. There exist $x \in M$ and $\alpha, \beta \in \Gamma$ such that $a\alpha\beta a = a$, $x\beta a\alpha x = x$, $b\alpha\beta b = b$ and $x\beta b\alpha x = x$. Then $(a\rho_1)\alpha(x\rho_1)\beta(a\rho_1) = a\rho_1, (x\rho_1)\beta(a\rho_1)\alpha(x\rho_1) = x\rho_1, (b\rho_1)\alpha(x\rho_1)\beta(b\rho_1) = b\rho_1, (x\rho_1)\beta(b\rho_1)\alpha(x\rho_1) = x\rho_1$. Hence $a\rho_1, b\rho_1 \in V_{\beta}^{\alpha}(x\rho_1)$. But M/ρ_1 is an inverse Γ -semigroup. Hence $|V_{\beta}^{\alpha}(x\rho_1)| = 1$. Then $a\rho_1 = b\rho_1$, so that $(a, b) \in \rho_1$. Hence $\rho \subset \rho_1$. This completes the proof.

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