FUNDAMENTAL THEOREM OF WIENER CALCULUS

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ABSTRACT. In this paper we define and develop a theory of differentiation in Wiener space $C[0,T]$. We then proceed to establish a fundamental theorem of the integral calculus for $C[0,T]$. First of all, we show that the derivative of the indefinite Wiener integral exists and equals the integrand functional. Secondly, we show that certain functionals defined on $C[0,T]$ are equal to the indefinite integral of their Wiener derivative.

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1. INTRODUCTION.

Consider the Wiener measure space $(C[0,T], \mathcal{F}^*, m_w)$ where $C[0,T]$ is the space of all continuous functions x on $[0,T]$ vanishing at the origin. For each partition $\tau = \tau_n = \{t_1, \dots, t_n\}$ of $[0,T]$ with $0 = t_0 < t_1 < \dots < t_n = T$, let $X_\tau: C[0,T] \rightarrow \mathbb{R}^n$ be defined by $X_\tau(x) \equiv x(\tau) = (x(t_1), \dots, x(t_n))$. Let $\mathcal{B}^n$ be the σ -algebra of Borel sets in $\mathbb{R}^n$. Then a set of the type

$$I = \{x \in C[0, T] : X_\tau(x) \in B\} \equiv X_\tau^{-1}(B), \quad B \in \mathcal{B}^n$$

is called a Wiener interval (or a Borel cylinder). It is well known that

$$m_w(I) = \int_B K(\tau, \vec{\eta}) d\vec{\eta}, \quad (1.1)$$

where

$$K(\tau, \vec{\eta}) = \left\{ \prod_{j=1}^n 2\pi(t_j - t_{j-1}) \right\}^{-1/2} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(\eta_j - \eta_{j-1})^2}{t_j - t_{j-1}} \right\} \quad (1.2)$$

with $\vec{\eta} = (\eta_1, \dots, \eta_n)$, and $\eta_0 = 0$. The measure m_w is a probability measure defined on the algebra $\mathcal{J}$ of all Wiener intervals and m_w is extended to the Caratheodory extension $\mathcal{J}^*$ of $\mathcal{J}$. Let $\mathcal{J}_\tau$ be the σ -algebra generated by the set $\{X_\tau^{-1}(B) : B \in \mathcal{B}^n\}$ with τ fixed. Then, by the definition of conditional expectation, see Doob [1], Tucker [2] and Yeh [3], for each Wiener integrable function $F(x)$,

$$\begin{aligned} \mu_\tau(B) &\equiv \int_{X_\tau^{-1}(B)} F(x) m_w(dx) = \int_{X_\tau^{-1}(B)} E(F | \mathcal{J}_\tau) m_w(dx) \\ &= \int_B E(F(x) | X_\tau(x) = \vec{\eta}) P_{X_\tau}(d\vec{\eta}), \quad B \in \mathcal{B}^n, \end{aligned} \quad (1.3)$$

where $P_{X_\tau}(B) = m_w(X_\tau^{-1}(B))$, and $E(F(x) | X_\tau(x) = \vec{\eta})$ is a Lebesgue measurable function of $\vec{\eta}$ which is unique up to null sets in $\mathbb{R}^n$. Also, using (1.1) and (1.3) and choosing $F(x) \equiv 1$, we see that

$$P_{X_\tau}(d\vec{\eta}) = K(\tau, \vec{\eta}) d\vec{\eta}, \quad (1.4)$$

or

$$\frac{dP_{X_\tau}}{d\vec{\eta}} = K(\tau, \vec{\eta}), \quad \vec{\eta} \in \mathbb{R}^n. \quad (1.5)$$

Next, for each $F \in L_1(C[0, T], m_w)$ and each partition τ of $C[0, T]$, let

$$F_\tau = E(F | \mathcal{J}_\tau) \quad (1.6)$$

and

$$\tilde{F}(\vec{\eta}) = E(F(x) | X_\tau(x) = \vec{\eta}) \equiv E(F | X_\tau)(\vec{\eta}). \quad (1.7)$$

Then, $\{F_\tau\}$ is a martingale, and by the martingale convergence theorem,

$$\lim_{\|\tau\| \rightarrow 0} F_\tau(x) = F(x) \quad (1.8)$$

for almost all $x \in C[0, T]$. Furthermore,

$$F(x) = \lim_{\|\tau\| \rightarrow 0} E(F(y) | X_\tau(y) = x(\tau)) = \lim_{\|\tau\| \rightarrow 0} \tilde{F}(x(\tau)) \quad (1.9)$$

for almost all $x \in C[0, T]$.

For a given partition $\tau = \tau_n$ of $[0, T]$ and $x \in C[0, T]$, define the polygonal function $[x] \equiv [x(\tau)]$ on $[0, T]$ by

$$[x](t) = x(t_{j-1}) + \frac{t-t_{j-1}}{t_j-t_{j-1}} (x(t_j) - x(t_{j-1})), \quad t_{j-1} \leq t \leq t_j, \quad j = 1, \dots, n.$$

Similarly, for each $\vec{\eta} = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$, define the polygonal function $[\vec{\eta}]$ of $\vec{\eta}$ on $[0, T]$ by

$$[\vec{\eta}](t) = \eta_{j-1} + \frac{t-t_{j-1}}{t_j-t_{j-1}} (\eta_j - \eta_{j-1}), \quad t_{j-1} \leq t \leq t_j, \quad j = 1, \dots, n \text{ with } \eta_0 = 0.$$

Then both functions $[x]$ and $[\vec{\eta}]$ are continuous on $[0, T]$, their graphs are line segments on each subinterval $[t_{j-1}, t_j]$, and $[x](t_j) = x(t_j)$ and $[\vec{\eta}](t_j) = \eta_j$ at each $t_j \in \tau$.

For $x, y \in C[0, T]$, we use the convention:

$$x \leq y \text{ if and only if } x(t) \leq y(t) \text{ for every } t \in [0, T],$$

and

$$x < y \text{ if and only if } x(t) < y(t) \text{ for every } t \in (0, T).$$

The main purpose of this paper is to define and develop a theory of differentiation in Wiener space $C[0, T]$, and then to establish a fundamental theorem of the integral calculus on $C[0, T]$; namely, that the Wiener derivative of the indefinite integral $\int_{y \leq x} F(y) m_w(dy)$ is $F(x)$, and that a Wiener absolutely continuous function can be expressed as the indefinite integral of its Wiener derivative. This study was initiated by Smolowitz [4]. In this paper we incorporate some recent results of Park and Skoug [5] to improve and substantially simplify the concepts and results of Smolowitz [4].

2. THE WIENER DERIVATIVE.

Our first objective is to define the Wiener derivative $\mathcal{D}_x(\cdot)$ so that

$$\mathcal{D}_x \int_{y \leq x} F(y) m_w(dy) = F(x)$$

for $F \in L_1(C[0, T], m_w)$. We start by quoting the following theorem from Park and Skoug [5] which plays an important role in this paper.

THEOREM A. Let $F \in L_1(C[0, T], m_w)$. Then for any Borel set $B \in \mathcal{B}^n$,

$$\mu_\tau(B) \equiv \int_{X_\tau^{-1}(B)} F(x) m_w(dx) = \int_B E_x[F(x) - [x] + [\vec{\eta}]] P_{X_\tau}(d\vec{\eta}) \quad (2.1)$$

where

$$E_x[F(x - [x] + [\vec{\eta}])] = \int_{C[0, T]} F(x - [x] + [\vec{\eta}]) m_w(dx).$$

In view of (1.3) and (2.1), we may conclude that

$$E(F(x) | X_\tau(x) = \vec{\eta}) = E_x[F(x - [x] + [\vec{\eta}])] \quad (2.2)$$

for almost all $\vec{\eta}$ in $\mathbb{R}^n$; i.e., we may express the conditional expectation $E(F | X_\tau)(\vec{\eta})$ in terms of an ordinary Wiener integral. Note that for $F \in L_1(C[0, T], m_w)$,

$\tilde{F}(\vec{\eta}) \equiv E(F | X_\tau)(\vec{\eta})$ is in $L_1(\mathbb{R}^n, P_{X_\tau}(d\vec{\eta}))$. Also note that for each $x \in C[0, T]$ and each

partition $\tau = \{t_1, \dots, t_n\}$ of $[0, T]$, $\tilde{F}(x(\tau)) = E(F(y) | X_\tau(y) = x(\tau))$ is a function of $x(t_1), \dots, x(t_n)$.

DEFINITION 1. Let $F \in L_1(C[0, T], m_w)$. For each partition $\tau = \{t_1, \dots, t_n\}$ of $[0, T]$ define the operator $\mathcal{D}_{x(\tau)}$ by

$$\mathcal{D}_{x(\tau)} F(x) = \frac{\partial^n \tilde{F}(x(\tau))}{\partial x(t_n) \cdots \partial x(t_1)} / K(\tau, x(\tau)) \quad (2.3)$$

if it exists. Furthermore, if $\mathcal{D}_{x(\tau)} F(x)$ exists for each partition τ , then the Wiener derivative of $F(x)$ is defined by

$$\mathcal{D}_x F(x) = \lim_{\|\tau\| \rightarrow 0} \mathcal{D}_{x(\tau)} F(x)$$

if the limit exists.

Our first theorem is the first half of the fundamental theorem of Wiener calculus.

THEOREM 1. Let $F \in L_1(C[0, T], m_w)$. Then

$$\mathcal{D}_x \int_{y \leq x} F(y) m_w(dy) = F(x)$$

for almost all $x \in C[0, T]$.

PROOF. For $x \in C[0, T]$ let $G(x)$ denote the indefinite Wiener integral

$$G(x) = \int_{y \leq x} F(y) m_w(dy) = E_y[I_x(y) F(y)] \quad (2.4)$$

where $I_x(y)$ is the indicator function

$$I_x(y) = \begin{cases} 1 & , \quad y(t) \leq x(t) \text{ for all } t \in [0, T] \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Then using (1.7), (2.4), (2.2), (1.3), (2.2) and the Fubini theorem, we obtain

$$\begin{aligned} \tilde{G}(\vec{\eta}) &= E(G(u) | X_\tau(u) = \vec{\eta}) \\ &= E_u(E_y[I_u(y) F(y)] | X_\tau(u) = \vec{\eta}) \\ &= E_u[E_y[I_{u-[u]+\lceil \vec{\eta} \rceil}(y) F(y)]] \\ &= E_u \left[\int_{\mathbb{R}^n} E_y(I_{u-[u]+\lceil \vec{\eta} \rceil}(y) F(y) | X_\tau(y) = \vec{\xi}) P_{X_\tau}(d\vec{\xi}) \right] \\ &= \int_{\mathbb{R}^n} E_u[E_y[I_{u-[u]+\lceil \vec{\eta} \rceil}(y-[y]+\lceil \vec{\xi} \rceil) F(y-[y]+\lceil \vec{\xi} \rceil)] P_{X_\tau}(d\vec{\xi}) . \end{aligned} \quad (2.5)$$

But $I_{u-[u]+\lceil \vec{\eta} \rceil}(y-[y]+\lceil \vec{\xi} \rceil)$ is zero unless

$$y(t) - [y](t) + \lceil \vec{\xi} \rceil(t) \leq u(t) - [u](t) + \lceil \vec{\eta} \rceil(t) \quad (2.6)$$

for all $t \in [0, T]$. But (2.6) implies that

$$\xi_j = y(t_j) - [y](t_j) + \lceil \vec{\xi} \rceil(t_j) \leq u(t_j) - [u](t_j) + \lceil \vec{\eta} \rceil(t_j) = \eta_j$$

for $j = 1, \dots, n$. Hence we can write

$$\begin{aligned} \tilde{G}(\vec{\eta}) = & \int_{-\infty}^{\eta_n} \cdots \int_{-\infty}^{\eta_1} E_u[E_y[I_{u-[u]+[\vec{\eta}]}(y-[y]+[\vec{\xi}]) \\ & \cdot F(y-[y]+[\vec{\xi}])]K(\tau, \vec{\xi})d\xi_1 \cdots d\xi_n, \end{aligned}$$

and so for each $x \in C[0, T]$,

$$\begin{aligned} \tilde{G}(x(\tau)) = & \int_{-\infty}^{x(t_n)} \cdots \int_{-\infty}^{x(t_1)} E_u[E_y[I_{u-[u]+[x(\tau)]}(y-[y]+[\vec{\xi}]) \\ & \cdot F(y-[y]+[\vec{\xi}])]K(\tau, \vec{\xi})d\xi_1 \cdots d\xi_n. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial^n \tilde{G}(x(\tau))}{\partial x(t_n) \cdots \partial x(t_1)} &= E_u[E_y[I_{u-[u]+[x(\tau)]}(y-[y]+[x(t)]) \\ & \cdot F(y-[y]+[x(t)])]K(\tau, x(\tau)) \\ &= E_{u,y}(I_u(y)F(y)|X_\tau(y) = x(\tau), X_\tau(u) = x(\tau))K(\tau, x(\tau)). \end{aligned} \quad (2.7)$$

Applying (1.9) to (2.7) yields

$$\begin{aligned} \mathcal{D}_x(G(x)) &= \lim_{\|\tau\| \rightarrow 0} E_{u,y}(I_u(y)F(y)|X_\tau(y) = x(\tau), X_\tau(u) = x(\tau)) \\ &= F(x) \end{aligned} \quad (2.8)$$

for almost all x in $C[0, T]$ which concludes the proof of Theorem 1.

COROLLARY 1. If $\{t'_1, \dots, t'_m\} \subseteq \tau = \{t_1, \dots, t_n\}$ and if $F(y) = f(y(t'_1), \dots, y(t'_m))$ is in $L_1(C[0, T], m_w)$, then

$$\mathcal{D}_{x(\tau)} \int_{y \leq x} F(y) m_w(dy) = F(x) E_{u,y}(I_u(y)|X_\tau(y) = X_\tau(u) = x(\tau))$$

and

$$\mathcal{D}_x \int_{y \leq x} F(y) m_w(dy) = F(x) = f(x(t'_1), \dots, x(t'_m))$$

for almost all x in $C[0, T]$.

PROOF. Using (2.7) and (2.4) we see that

$$\mathcal{D}_{x(\tau)} \int_{y \leq x} F(y) m_w(dy) = E_{u,y}(I_u(y)F(y)|X_\tau(y) = x(\tau), X_\tau(u) = x(\tau)).$$

Under the conditioning $X_\tau(y) = x(\tau)$, $F(y)$ becomes $f(x(t'_1), \dots, x(t'_m))$ which equals $F(x)$.

Therefore,

$$\mathcal{D}_{x(\tau)} \int_{y \leq x} F(y) m_w(dy) = F(x) E_{u,y}(I_u(y)|X_\tau(y) = X_\tau(u) = x(\tau)).$$

As $\|\tau\| \rightarrow 0$, $E_{u,y}(I_u(y)|X_\tau(y) = X_\tau(u) = x(\tau)) \rightarrow I_x(x) = 1$ by (1.9) for almost all x in $C[0, T]$. Thus Corollary 1 is established.

COROLLARY 2. Let $\tau' = \{t'_1, \dots, t'_m\}$ be any partition of $[0, T]$, and let $F(x) = f(x(t'_1), \dots, x(t'_m))$ be in $L_1(C[0, T], m_w)$. Then $\mathcal{D}_x F(x) = 0$.

PROOF. Let τ be a partition of $[0, T]$ properly containing τ' . Then

$$\tilde{F}(x(\tau)) = E(F|y)|X_\tau(y) = x(\tau)) = f(x(t'_1), \dots, x(t'_n)).$$

Thus $\mathcal{D}_{x(\tau)}F(x) = 0$, and so $\mathcal{D}_x F(x) = 0$.

3. LEBESGUE AND WIENER ABSOLUTE CONTINUITY.

In this section we show that certain functions defined on $C[0, T]$ are equal to the indefinite integral of their Wiener derivative.

For $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$ in $\mathbb{R}^n$ with $a_i < b_i$, $i = 1, \dots, n$, let $V(\vec{a}, \vec{b}, k)$ be the collection of all points of the form $\vec{v} = (v_1, \dots, v_n)$ where each v_i is either a_i or b_i and exactly k of the v_i are a_i 's. For any function f defined on $V(\vec{a}, \vec{b}, k)$ for $k = 0, 1, \dots, n$, let

$$\Delta_{\vec{a}, \vec{b}} f = f(\vec{b}) + \sum_{k=1}^n (-1)^k \sum_{\vec{v} \in V(\vec{a}, \vec{b}, k)} f(\vec{v}). \quad (3.1)$$

A function of n variables $f(u_1, \dots, u_n)$ is said to be Lebesgue absolutely continuous in the sense of Vitali (see Clarkson and Adams [6, 7] and Hobson [8]) on the region $\Omega \subset \mathbb{R}^n$ if,

given $\varepsilon > 0$, there is a $\delta > 0$ such that if $I_k^* \equiv \times_{i=1}^n (\alpha_i^{(k)}, \beta_i^{(k)})$, $k = 1, 2, \dots$, are disjoint

n -dimensional rectangles contained in Ω with $\sum_{k=1}^N m_L(I_k^*) < \delta$ for any N , then

$\sum_{k=1}^N |\Delta_{\vec{\alpha}^{(k)}, \vec{\beta}^{(k)}} f| < \varepsilon$, where $m_L(\cdot)$ denotes n -dimensional Lebesgue measure, and

$\vec{\alpha}^{(k)} = (\alpha_1^{(k)}, \dots, \alpha_n^{(k)})$. A function $f(u_1, \dots, u_n)$ is said to be Lebesgue absolutely

continuous (in the sense of Hardy-Krause; see Berkson and Gillespie [9], and Clarkson and Adams [6, 7]) on a region $\Omega \subset \mathbb{R}^n$ if for each $k = 1, \dots, n-1$, whenever $n-k$ variables are fixed then f , as a function of its remaining k variables, is Lebesgue absolutely continuous in the sense of Vitali on $\Omega \cap \mathbb{R}^k$. When we merely state "Lebesgue absolutely continuous", it is always meant in the sense of Hardy-Krause.

It is well known that if $f(u_1, \dots, u_n)$ is Lebesgue absolutely continuous in the sense of Vitali on $R \equiv \times_{i=1}^n [a_i, b_i]$, then $\partial^n f(u_1, \dots, u_n) / \partial u_1 \cdots \partial u_n$ exists a.e. on R and is integrable on R . Furthermore

$$\int_R [\partial^n f(u_1, \dots, u_n) / \partial u_1 \cdots \partial u_n] du_1 \cdots du_n = \Delta_{\vec{a}, \vec{b}} f, \quad (3.2)$$

and

$$\int_R |\partial^n f(u_1, \dots, u_n) / \partial u_1 \cdots \partial u_n| du_1 \cdots du_n = \text{Var}(f, R) \quad (3.3)$$

where $\text{Var}(f, R)$ denotes the total variation of f over R .

Let $G(x)$ be any Wiener integrable function on $C[0, T]$. Then, by definition,

$$\begin{aligned}\tilde{G}(\vec{\eta}) &= E(G|X_\tau)(\vec{\eta}) \text{ is a function of } \vec{\eta} \text{ which is integrable with respect to} \\ P_{X_\tau}(d\vec{\eta}) &= K(\tau, \vec{\eta})d\vec{\eta}.\end{aligned}$$

DEFINITION 2. A Wiener integrable function $G(x)$ defined on $C[0, T]$ is said to be Wiener absolutely continuous provided that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if the sequence $I_k \equiv \{x \in C[0, T] : \alpha_i^{(k)} < x(s_i^{(k)}) \leq \beta_i^{(k)}, i = 1, \dots, m^{(k)}\}$ with $-\infty \leq \alpha_i^{(k)} < \beta_i^{(k)} \leq \infty$ are disjoint Wiener intervals with $\sum_{k=1}^N m_w(I_k) < \delta$ for any N , then $\sum_{k=1}^N |\Delta_{\vec{\alpha}^{(k)}, \vec{\beta}^{(k)}} \tilde{G}| < \varepsilon$.

The following propositions can be easily established.

PROPOSITION A. If $G(x)$ is Wiener absolutely continuous on $C[0, T]$, then for every partition τ of $(0, T]$, $\tilde{G}(\vec{\eta})$ is Lebesgue absolutely continuous on $\mathbb{R}^{|\tau|}$, where $|\tau|$ denotes the number of points in τ .

PROPOSITION B. Let $F \in L_1(C[0, T], m_w)$. Then the indefinite Wiener integral $G(x) = \int_{y \leq x} F(y) m_w(dy)$ is Wiener absolutely continuous on $C[0, T]$.

Our next theorem is the second half of the fundamental theorem of Wiener Calculus.

THEOREM 2. Let $G \in L_1(C[0, T], m_w)$ satisfy the conditions:

- (i) $\mathcal{D}_x G(x)$ exists for almost all $x \in C[0, T]$ and belongs to $L_1(C[0, T], m_w)$,
- (ii) $G(x)$ is Wiener absolutely continuous on $C[0, T]$,
- (iii) If $\{x_k\}_{k=1}^\infty$ is a sequence in $C[0, T]$ such that $x_k(s_0) \rightarrow -\infty$ as $k \rightarrow \infty$ for some fixed point $s_0 \in (0, T]$, then $G(x_k) \rightarrow 0$ as $k \rightarrow \infty$.

Then

$$G(u) = \int_{x \leq u} \mathcal{D}_x G(x) m_w(dx) \quad (3.4)$$

for almost all u in $C[0, T]$.

PROOF. For given $\varepsilon > 0$ let $\delta = \delta(\varepsilon/3) > 0$ be the value for the Wiener absolute continuity of $G(x)$, and also assume that

$$m_w(S) < \delta \Rightarrow \int_S |\mathcal{D}_x G(x)| m_w(dx) < \varepsilon/3.$$

Let $\{\tau^{(k)}\}$ be a sequence of partitions of $[0, T]$ such that $\|\tau^{(k)}\| \rightarrow 0$ as $k \rightarrow \infty$. Then

$$\lim_{k \rightarrow \infty} \mathcal{D}_{x(\tau^{(k)})} G(x) = \mathcal{D}_x G(x) \text{ for almost all } x \in C[0, T].$$

By Egoroff's theorem, there exists a set $C_\varepsilon \subset C[0, T]$ with $m_w(C_\varepsilon) > 1 - \delta/2$ and a positive integer k_0 such that if $k \geq k_0$, then

$$|\mathcal{D}_{x(\tau^{(k)})} G(x) - \mathcal{D}_x G(x)| < \varepsilon/3 \text{ for every } x \in C_\varepsilon.$$

Let

$$C_k = \{x \in C[0, T] : |\mathcal{D}_{x(\tau^{(k)})} G(x) - \mathcal{D}_x G(x)| < \varepsilon/3\}, \quad k \geq k_0.$$

Then $C_\varepsilon \subset C_k$, and hence $m_w(C_k) \geq m_w(C_\varepsilon) > 1 - \delta/2$, and

$$\int_{C_k} |\mathcal{D}_{x(\tau^{(k)})} G(x) - \mathcal{D}_x G(x)| m_w(dx) < \varepsilon/3 \text{ for } k \geq k_0.$$

The complements satisfy $m_w(C_k^c) \leq m_w(C_\varepsilon^c) < \delta/2$ for $k \geq k_0$. Next consider fixed k ,

$k \geq k_0$ and let q denote the number of points in the partition $\tau^{(k)}$. Let

$$E_k \equiv \{\vec{\eta} = (\eta_1, \dots, \eta_q) \in \mathbb{R}^q : \vec{\eta} = x(\tau^{(k)}) \text{ for some } x \in C_k^c\}.$$

Then,

$$m_w(C_k^c) = \int_{E_k} K(\tau^{(k)}, \vec{\eta}) d\vec{\eta}.$$

Since $K(\tau^{(k)}, \vec{\eta})$ is bounded in $\vec{\eta}$ on $\mathbb{R}^q$ and $\int_{E_k} K(\tau^{(k)}, \vec{\eta}) d\vec{\eta} = m_w(C_k^c) < \delta/2$, we can find

a countable sequence of disjoint q -dimensional rectangles $I_\ell^* = \prod_{i=1}^q (\alpha_i^{(\ell)}, \beta_i^{(\ell)})$, $\ell = 1, 2, \dots$

such that $E_k \subset \bigcup_{\ell=1}^{\infty} I_\ell^*$, and

$$\int_{E_k} K(\tau^{(k)}, \vec{\eta}) d\vec{\eta} \leq \sum_{\ell=1}^{\infty} \int_{I_\ell^*} K(\tau^{(k)}, \vec{\eta}) d\vec{\eta} = \sum_{\ell=1}^{\infty} m_w(I_\ell^*) < \delta,$$

where

$$I_\ell = \{x \in C[0, T] : \alpha_i^{(\ell)} < x(s_i^{(k)}) \leq \beta_i^{(\ell)} \text{ for each } s_i^{(k)} \in \tau^{(k)}\}.$$

Hence

$$\begin{aligned} \int_{C_k^c} |\mathcal{D}_{x(\tau^{(k)})} G(x)| m_w(dx) &= \int_{E_k} |\partial^q \tilde{G}(\eta_1, \dots, \eta_q) / \partial \eta_1 \cdots \partial \eta_q| d\eta_1 \cdots d\eta_q \\ &\leq \sum_{\ell=1}^{\infty} \int_{\alpha_q^{(\ell)}}^{\beta_q^{(\ell)}} \cdots \int_{\alpha_1^{(\ell)}}^{\beta_1^{(\ell)}} |\partial^q \tilde{G}(\eta_1, \dots, \eta_q) / \partial \eta_1 \cdots \partial \eta_q| d\eta_1 \cdots d\eta_q \\ &= \sum_{\ell=1}^{\infty} \text{Var}(\tilde{G}, I_\ell^*), \end{aligned}$$

where the last equality follows from (3.3). Now,

$$\text{Var}(\tilde{G}, I_\ell^*) = \sup_i \sum_{\vec{\sigma}_i, \vec{\rho}_i} |\Delta_{\vec{\sigma}_i, \vec{\rho}_i} \tilde{G}(\vec{\eta})|$$

where the supremum is taken over all possible nets of I_ℓ^* , and each net has total Lebesgue

measure equal to that of I_ℓ^* , and so the corresponding Wiener intervals have total measure equal to $m_w(I_\ell)$. Since $\sum_{\ell=1}^{\infty} m_w(I_\ell) < \delta(\varepsilon/3)$, by the Wiener absolute continuity of G , we have

$$\sum_{\ell=1}^N \sum_i |\Delta_{\vec{\sigma}_i, \vec{\rho}_i} \tilde{G}(\vec{\eta})| < \varepsilon/3 \text{ for every } N \text{ and every net.}$$

Thus, by taking the supremum over all nets, we get

$$\sum_{\ell=1}^N \text{Var}(\tilde{G}, I_\ell^*) \leq \varepsilon/3 \text{ for every } N,$$

and hence

$$\int_{C_k} |\mathcal{D}_{x(\tau(k))} G(x)| m_w(dx) \leq \varepsilon/3.$$

Thus, for every $k \geq k_0$,

$$\begin{aligned} & \int_{C[0,T]} |\mathcal{D}_{x(\tau(k))} G(x) - \mathcal{D}_x G(x)| m_w(dx) \\ & \leq \int_{C_k} |\mathcal{D}_{x(\tau(k))} G(x) - \mathcal{D}_x G(x)| m_w(dx) \\ & \quad + \int_{C_k} |\mathcal{D}_{x(\tau(k))} G(x)| m_w(dx) + \int_{C_k} |\mathcal{D}_x G(x)| m_w(dx) \\ & < \varepsilon. \end{aligned}$$

In particular

$$\int_{[x] \leq [u]} |\mathcal{D}_{x(\tau(k))} G(x) - \mathcal{D}_x G(x)| m_w(dx) < \varepsilon \text{ for } k \geq k_0,$$

where $[\cdot]$ corresponds to $\tau^{(k)}$. Hence

$$\lim_{k \rightarrow \infty} \left[\int_{[x] \leq [u]} \mathcal{D}_{x(\tau(k))} G(x) m_w(dx) - \int_{[x] \leq [u]} \mathcal{D}_x G(x) m_w(dx) \right] = 0.$$

Since $\{x \in C[0,T] : [x] \leq [u]\} \rightarrow \{x \in C[0,T] : x \leq u\}$ as $k \rightarrow \infty$, an application of the dominated convergence theorem yields

$$\lim_{k \rightarrow \infty} \int_{[x] \leq [u]} \mathcal{D}_x G(x) m_w(dx) = \int_{x \leq u} \mathcal{D}_x G(x) m_w(dx).$$

Thus,

$$\lim_{k \rightarrow \infty} \int_{[x] \leq [u]} \mathcal{D}_{x(\tau(k))} G(x) m_w(dx) = \int_{x \leq u} \mathcal{D}_x G(x) m_w(dx). \quad (3.5)$$

On the other hand, using (2.3), (3.2) and (3.1), we see that for any $a \in C[0,T]$ with

$a < u$,

$$\begin{aligned}
& \int_{[a] \leq [x] \leq [u]} \mathcal{D}_{x(\tau^{(k)})} G(x) m_w(dx) \\
&= \int_{a(\tau^{(k)}) \leq x(\tau^{(k)}) \leq u(\tau^{(k)})} \mathcal{D}_{x(\tau^{(k)})} G(x) m_w(dx) \\
&= \int_{a(s_q^{(k)})}^{u(s_q^{(k)})} \cdots \int_{a(s_1^{(k)})}^{u(s_1^{(k)})} [\partial^q \tilde{G}(\eta_1, \dots, \eta_q) / \partial \eta_1 \cdots \partial \eta_q] d\eta_1 \cdots d\eta_q \quad (3.6) \\
&= \Delta_{a(\tau^{(k)}), u(\tau^{(k)})} \tilde{G} \\
&= \tilde{G}(u(\tau^{(k)})) + \sum_{\ell=1}^q (-1)^\ell \sum_{\vec{v} \in V(a(\tau^{(k)}), u(\tau^{(k)}), \ell)} \tilde{G}(\vec{v}) .
\end{aligned}$$

If we let $a(s_i^{(k)}) \rightarrow -\infty$ as $k \rightarrow \infty$ for $i = 1, \dots, q$ in (3.6), then by assumption (iii), $\tilde{G}(\vec{v}) \rightarrow 0$ as $k \rightarrow \infty$ for every $\vec{v} \in V(a(\tau^{(k)}), u(\tau^{(k)}), \ell)$, $\ell \geq 1$. Thus (3.6) reduces to

$$\int_{[x] \leq [u]} \mathcal{D}_{x(\tau^{(k)})} G(x) m_w(dx) = \tilde{G}(u(\tau^{(k)})) . \quad (3.7)$$

In view of (1.9) and (3.5), we conclude that

$$\int_{x \leq u} \mathcal{D}_x G(x) m_w(dx) = G(u) \quad \text{for almost all } u \text{ in } C[0, T],$$

and so (3.4) is established.

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