OSCILLATION AND NONOSCILLATION IN NONLINEAR THIRD ORDER DIFFERENCE EQUATIONS

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(Received August 18, 1988 and in revised form June 20, 1989)

KEY WORDS AND PHRASES. Difference equations, third order, oscillatory and nonoscillatory solutions.

1980 AMS SUBJECT CLASSIFICATION CODES. 39A10, 39A12.

1. INTRODUCTION.

This paper is concerned with the oscillatory behavior of solutions of the third order nonlinear difference equation

$$\Delta(P_n \Delta^2 V_n) + Q_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) = 0, \quad n = 1, 2, \dots, \quad (1.1)$$

where Δ is the forward difference operator i.e., $\Delta V_n = V_{n+1} - V_n$. It will be assumed throughout that the conditions below are satisfied:

$$(I) \quad P_n > 0, \Delta P_n > 0 \text{ and } Q_n > 0 \text{ for } n = 0, 1, 2, \dots$$

$$(II) \quad \sum_{n=1}^{\infty} \frac{1}{P_n} = \infty$$

$$(III) \quad f: \mathbb{R}^3 \rightarrow \mathbb{R} \text{ is continuous and } xf(x, y, z) > 0 \text{ for } x \neq 0.$$

By a solution of (1.1) we mean a real sequence V satisfying equation (1.1) for $n = 1, 2, \dots$. A solution V of (1.1) is called nonoscillatory if it is eventually positive or eventually negative. Otherwise, it is called oscillatory. The problem of determining oscillation criteria for certain second order nonlinear difference equations has been investigated by Hooker and Patula [1], and Szmanda [2]. The results of [2] were generalized by Li [3]. This paper examines a slightly more general equation than those studied in [2] and [3]. The authors began a study of similar equations in [4].

2. MAIN RESULTS.

LEMMA 2.1. Suppose V is a nonoscillatory solution of (1.1). Then, either

$$\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n \quad (2.1)$$

for all n sufficiently large, or

$$\operatorname{sgn} V_n = \operatorname{sgn} \Delta^2 V_n \neq \operatorname{sgn} \Delta V_n \quad (2.2)$$

for all sufficiently large n , and $\lim_{n \rightarrow \infty} \Delta V_n = \lim_{n \rightarrow \infty} \Delta^2 V_n = 0$.

PROOF. Assume V is a nonoscillatory solution of (1.1), where $V_n > 0$ for all $n > N$, where N is a positive integer. The proof is similar if $V_n < 0$ for all $n > N$. Note that $\Delta(P_n \Delta^2 V_n) = -Q_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) < 0$, for each $n > N$. Thus $P_n \Delta^2 V_n$ is decreasing and is eventually sign definite. A positive integer $M > N$ exists for which ΔV_n and $\Delta^2 V_n$ are of one sign when $n > M$. The following cases must be considered:

(a) $V_n > 0$, $\Delta V_n > 0$, $\Delta^2 V_n > 0$, $n > M$,

(b) $V_n > 0$, $\Delta V_n < 0$, $\Delta^2 V_n > 0$, $n > M$,

(c) $V_n > 0$, $\Delta V_n < 0$, $\Delta^2 V_n < 0$, $n > M$,

(d) $V_n > 0$, $\Delta V_n > 0$, $\Delta^2 V_n < 0$, $n > M$.

Case (c) is impossible because if $\Delta V_n \Delta^2 V_n > 0$ for all sufficiently large n , then $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n$ eventually. We show that (d) is also impossible. If (d) holds, then from above $P_n \Delta^2 V_n$ is negative and decreasing for all n sufficiently large. Let $k < 0$ be such that $P_n \Delta^2 V_n < k$ for all $n > M$. Then $\Delta^2 V_n < \frac{k}{P_n}$, $n > M$. Summing from M to $R-1$ we obtain

$$\Delta V_R - \Delta V_M < k \sum_{n=M}^{R-1} \frac{1}{P_n}.$$

Letting $R \rightarrow \infty$, implies ΔV_R is eventually negative, but this contradicts (d), therefore (d) cannot hold. This completes the proof of the lemma.

We continue our study of (1.1) by considering a functional which plays a vital role in our investigation. Similar functionals have been used to study solutions of differential equations (Taylor [5]).

LEMMA 2.2. Let V be a solution of (1.1). Then

$$F[V_n] = F_n = 2V_n P_n \Delta^2 V_n - P_{n-1} (\Delta V_n)^2$$

is nonincreasing, in fact

$$\Delta F_n = -2Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) - P_n (\Delta^2 V_n)^2 - \Delta P_{n-1} (\Delta V_n)^2.$$

Since F_n is monotone for any nontrivial solution of (1.1) we have that F_n is of one sign for all n sufficiently large. Using this we will examine solutions of (1.1) where $F_n > 0$ for each n and those for which $F_m < 0$ for some positive integer m .

THEOREM 2.1. Let V be a nontrivial solution of (1.1) for which $F[V_n] > 0$. Then the following are true:

$$(i) \quad \sum_{n=0}^{\infty} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) < \infty,$$

$$(ii) \quad \sum_{n=0}^{\infty} P_n (\Delta^2 V_n)^2 < \infty, \text{ and}$$

$$(iii) \quad \sum_{n=0}^{\infty} \Delta P_{n-1} (\Delta V_n)^2 < \infty.$$

PROOF. Since $F_n > 0$ for each n , differencing F_n and summing from 0 to $k-1$ we find

$$0 < F_k = F_0 - 2 \sum_{n=0}^{k-1} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) - \sum_{n=0}^{k-1} P_n (\Delta^2 V_n)^2 - \sum_{n=0}^{k-1} \Delta P_{n-1} (\Delta V_n)^2.$$

Thus,

$$2 \sum_{n=0}^{k-1} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) + \sum_{n=0}^{k-1} P_n (\Delta^2 V_n)^2 + \sum_{n=0}^{k-1} \Delta P_{n-1} (\Delta V_n)^2 < F_0.$$

Allowing k to tend to infinity establishes each of (i), (ii) and (iii) since F_0 is independent of k .

THEOREM 2.2. Suppose that $\frac{f(x,y,z)}{x} > r > 0$ for $x \neq 0$ and $\liminf Q_n > 0$. Let V be a solution of (1.1) for which $F[V_n] > 0$ for each n . Then

$$(iv) \quad \sum_{n=0}^{\infty} V_n^2 < \infty,$$

$$(v) \quad \lim_{n \rightarrow \infty} V_n = \lim_{n \rightarrow \infty} \Delta V_n = \lim_{n \rightarrow \infty} \Delta^2 V_n = 0.$$

PROOF. To prove (iv), observe that

$$V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) > r V_{n+1}^2.$$

Thus

$$Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) > \alpha V_{n+1}^2$$

where $\alpha = \liminf Q_n$, so we have

$$\text{or } \sum_0^{\infty} V_{n+1}^2 < \sum_0^{\infty} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}).$$

Now apply (i), of Theorem 2.1, and the proof of (iv) is complete.

The relations (v) follow as a consequence of (iv).

EXAMPLE 2.1. As an illustration of Theorem 2.2 consider equation (1.1) with $P_n = n$,

$$Q_n = \frac{2^{2n}(n-1)}{2^{2n+2}+1}, \quad f(x, y, z) = x^3 + x.$$

Then

$$\Delta(n\Delta^2 V_n) + \frac{2^{2n}(n-1)}{2^{2n+2}+1} (V_{n+1}^3 + V_{n+1}) = 0.$$

The sequence defined by 2^{-n} is a solution of this equation for which $F_n \rightarrow 0$ as $n \rightarrow \infty$.

THEOREM 2.3. If $\frac{f(x, y, z)}{x} > r > 0$, and $\sum_0^{\infty} Q_n = \infty$, then every nonoscillatory solution of (1.1) approaches zero as $n \rightarrow \infty$.

PROOF. Suppose V is an eventually positive solution of (1.1) that is bounded away from zero, i.e. $V_n > \beta > 0$ for all n sufficiently large. Because of Lemma 2.1, an integer M exists so that the relations (2.1) or (2.2) are satisfied by V for all $n > M$. Now $f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) > rV_{n+1}$ where r is a positive constant. From (1.1) we find

$$\Delta(P_n \Delta^2 V_n) = -Q_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}).$$

Thus,

$$\Delta(P_n \Delta^2 V_n) < -rQ_n V_{n+1}. \quad (2.3)$$

Summing both sides of (2.3) from M to $k-1$ we find

$$P_k \Delta^2 V_k < P_M \Delta^2 V_M - r \sum_M^{k-1} Q_n V_{n+1} < P_M \Delta^2 V_M - r\beta \sum_M^{k-1} Q_n. \quad (2.4)$$

But as $k \rightarrow \infty$, the right hand side of (2.4) tends to $-\infty$, which in turn forces $P_k \Delta^2 V_k$ to tend to $-\infty$, and hence $\Delta^2 V_k < 0$ eventually, a contradiction of relations (2.1) and (2.2). A similar argument treats the case of an eventually negative solution. This completes the proof of the theorem.

COROLLARY 2.1. If $\frac{f(x, y, z)}{x} > r > 0$, for $x \neq 0$, and $\sum_0^{\infty} Q_n = \infty$, then every nonoscillatory solution of (1.1) satisfies the relations (2.2).

We are now in a position to show that oscillatory solutions exist under certain conditions.

THEOREM 2.4. Suppose $\frac{f(x, y, z)}{x} > r > 0$, for $x \neq 0$, $\sum_0^{\infty} Q_n = \infty$, and P_n is bounded. If V is a solution of (1.1) for which $F[V_n] < 0$ for some n , then V is oscillatory.

PROOF. Suppose V is a nonoscillatory solution of (1.1). We may suppose without any generality loss that $V_n > 0$ and $F[V_n] < 0$ for all $n > N$, since $F[V_n]$ is nonincreasing as $n \rightarrow \infty$. From Theorem 2.3, $V_n \rightarrow 0$, $\Delta V_n \rightarrow 0$ and $\Delta^2 V_n \rightarrow 0$ as $n \rightarrow \infty$.

This together with the boundedness of P_n implies that $F[V_n] \rightarrow 0$. This is clearly impossible since $F_n < 0$ and $\Delta F_n < 0$ for large n and the proof follows by contradiction.

Under certain conditions the bounded solutions of (1.1) behave rather nicely. Similar results appeared in [2] and [3].

THEOREM 2.5. Suppose $\sum_{n=0}^{\infty} nQ_n = \infty$ and $P_n < \beta$, β constant. Then every bounded solution of (1.1) is either oscillatory or tends to zero monotonically.

PROOF. By Lemma 2.1 a bounded nonoscillatory solution V satisfies

$$\operatorname{sgn} V_n = \operatorname{sgn} P_n \Delta^2 V_n \neq \operatorname{sgn} \Delta V_n$$

for all n sufficiently large. Assume that $V_n > 0$ eventually and suppose

$\lim_{n \rightarrow \infty} \Delta V_n = A_0$ where $A_0 > 0$. Note also

$$\lim_{n \rightarrow \infty} \Delta V_n = \lim_{n \rightarrow \infty} P_n \Delta^2 V_n = \lim_{n \rightarrow \infty} \Delta^2 V_n = 0.$$

The fact that $P_n \Delta^2 V_n \rightarrow 0$ as $n \rightarrow \infty$ follows from the boundedness of V_n and (II). Consider the sequence $r_n = n(P_n \Delta^2 V_n)$. Note that

$$\Delta r_n = P_{n+1} \Delta^2 V_{n+1} - nQ_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}). \quad (2.5)$$

Since f is continuous $f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) \rightarrow f(A_0, 0, 0) > 0$ as $n \rightarrow \infty$, so there exists N such that

$$f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) > \frac{1}{2} f(A_0, 0, 0) \text{ for all } n > N.$$

Therefore from (2.5) we have

$$\begin{aligned} \Delta r_n &< P_{n+1} \Delta^2 V_{n+1} - \frac{1}{2} n Q_n f(A_0, 0, 0) \\ &< \beta \Delta^2 V_{n+1} - \frac{1}{2} n Q_n f(A_0, 0, 0). \end{aligned}$$

Summing, from N to n

$$r_{n+1} < r_N + \beta \Delta V_{n+2} - \beta V_{N+1} - \frac{1}{2} f(A_0, 0, 0) \sum_{j=N}^n j Q_j.$$

As $n \rightarrow \infty$, $r_n \rightarrow -\infty$, a contradiction. Therefore $A_0 = 0$. This completes the proof of the theorem.

Finally we have

THEOREM 2.6. Suppose $\sum_{n=0}^{\infty} Q_n = \infty$, $f(\alpha x, \alpha y, \alpha z) = \alpha^{2m+1} f(x, y, z)$, $\alpha \neq 0$ and $f(x, y + h, z) > f(x, y, z)$ for $h > 0$. Then every solution of (1.) is either bounded or oscillatory.

PROOF. Suppose V_n is an unbounded nonoscillatory solution of (1.1). Without loss of generality we have

$$V_n > 0, \Delta V_n > 0, P_n \Delta^2 V_n > 0,$$

for all n sufficiently large. Consider the functional

$$q_n = \frac{P_n \Delta^2 V_n}{V_n}.$$

Differencing q_n we find

$$\begin{aligned} \Delta q_n &= \frac{-Q_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1})}{V_{n+1}} - \frac{P_n \Delta V_n \Delta^2 V_n}{V_n V_{n+1}} \\ &< -\frac{V_{n+1}^{2m}}{V_{n+1}} Q_n f(1, \frac{\Delta V_{n+1}}{V_{n+1}}, \frac{\Delta^2 V_{n+1}}{V_{n+1}}) \\ &< -\frac{V_{n+1}^{2m}}{V_{n+1}} Q_n f(1, 0, \frac{\Delta^2 V_{n+1}}{V_{n+1}}) \\ &< -\frac{V_N^{2m} Q_n f(1, 0, 0)}{2}. \end{aligned}$$

Summing we obtain

$$q_m < K_0 - \frac{V_N^{2m} f(1, 0, 0)}{2} \sum_N^{m-1} Q_n.$$

But this implies $q_m \rightarrow -\infty$ as $m \rightarrow \infty$, a contradiction since P_n , $\Delta^2 V_n$ and V_n are positive for all n sufficiently large.

EXAMPLE 2.2. It is possible for equations of the form of (1.1) to have unbounded oscillatory solutions. The sequence $V_n = (-2)^n$ is a solution of

$$\Delta^3 V_n + \frac{11}{18(4^{n+1})} (\Delta V_{n+1})^3 + 3V_{n+1} = 0.$$

Note that this example does not violate the conclusion of Theorem 2.6. Note also that (III) is not satisfied.

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