ULTRAREGULAR INDUCTIVE LIMITS

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ABSTRACT. An inductive limit \( E = \text{indlim} \ E_n \) is ultraregular if it is regular and each set \( B \subseteq E_n \), which is bounded in \( E \), is also bounded in \( E_n \). A necessary and sufficient condition for ultraregularity of \( E \) is given provided each \( E_n \) is an LF-space which is closed in \( E_{n+1} \).

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Let \( F \) be a locally convex space and \( B \subseteq F \) an absolutely convex subset. We denote by \( F_B \) the linear hull of \( B \) and provide it with the topology generated by the Minkowski functional of \( B \). If the topological space \( F_B \) is Banach then \( B \) is called a Banach disk. In [1] de Wilde calls the space \( F \) fast complete if every set \( B \), bounded in \( F \), is contained in a bounded Banach disk. Every sequentially complete space is fast complete.

A strict inductive limit of a sequence \( F_1 \subseteq F_2 \subseteq \cdots \) of Fréchet spaces is called an LF-space.

If \( S \subseteq X \cap Y \), where \( X \) and \( Y \) are locally convex spaces, then \( cl_X S \) and \( cl_Y S \) are the respective closures of \( S \) in \( X \) and \( Y \). Throughout the paper \( E_1 \subseteq E_2 \subseteq \cdots \) is a sequence of Hausdorff locally convex spaces with all inclusions id: \( E_n \rightarrow E_{n+1} \) continuous. We denote indlim \( E_n \) by \( E \).

In [2] Floret calls an inductive limit \( E \) regular, resp. \( \alpha \)-regular, if every set bounded in \( E \) is bounded, resp. contained, in some \( E_n \). An \( \alpha \)-regular inductive limit is ultraregular, resp. weakly ultraregular, if each set \( B \subseteq E_n \), which is bounded in \( E \), is also bounded, resp. weakly bounded, in \( E_n \).

In [3, §4, Prop. 4] Dieudonné and Schwartz proved that a strict inductive limit is ultraregular if each space \( E_n \) is closed in \( E_{n+1} \). In the case the inductive limit is not strict some restrictions on topologies of the spaces \( E_n \) have to be imposed.

We introduce two properties:

\((P)\) Every closed absolutely convex neighborhood in \( E_n \) is closed in \( E_{n+1} \).
If $U$ is a closed absolutely convex neighborhood in $E_n$ and $V$ is the closure of $U$ in $E_{n+1}$, then $U = V \cap E_n$.

Evidently $P \Rightarrow Q$ and if each space $E_n$ is closed in $E_{n+1}$ then $P$ is equivalent to $Q$. If the inductive limit $E$ is strict then $Q$ holds.

**Lemma 1.** $(Q)$ holds iff each real $f \in E'_n$ has a real continuous linear extension to $E_{n+1}$.

**Proof.** Assume $(Q)$ and take a real $f \in E'_n$, $f \neq 0$. The set $U = f^{-1}[-1,1]$ is a closed absolutely convex neighborhood in $E_n$. Let $V$ be the corresponding set $V \subset E_{n+1}$ from $(Q)$. If $E \subset V$ we would have $U = V \cap E_n = E_n$ and $f = 0$, which contradicts our assumption $f \neq 0$. Hence we can choose $x_0 \notin E \setminus V$. Since $V$ is absolutely convex and closed in $E_{n+1}$, there exists a real $g \in E'_{n+1}$ such that $V \subset g^{-1}(-\infty,1]$ and $g(x_0) > 1$. Further $f^{-1}(0) \subset g^{-1}(0)$ which implies $f(x_0) \neq 0$. Without loss of generality we may assume $f(x_0) = 1$. Then $f(z - f(z)x_0) = 0$ for $z \in E_n$ and $(z - f(z)x_0) \in f^{-1}(0) \subset g^{-1}(0)$. Hence, $g(z - f(z)x_0) = g(z) - f(z)g(x_0) = 0$ and the functional $(g(x_0))^{-1}g \in E'_{n+1}$ is the desired extension.

Assume that each real $f \in E'_n$ has a real extension $g \in E'_{n+1}$. Take a closed absolutely convex neighborhood $U \subset E_n$. By the Hahn-Banach theorem there exists $F \subset E'_n$ such that each $f \in F$ is real and $U = \{f^{-1}(-\infty,1]; f \in F\}$. Let $G$ be the set of all real extensions of all $f \in F$ to $E_{n+1}$. The set $V = \bigcap\{g^{-1}(-\infty,1]; g \in G\}$ is closed and absolutely convex in $E_{n+1}$. Evidently $U \subset V \cap E_n$. Assume $U \neq V \cap E_n$. Then there is $y \in (V \cap E_n) \setminus U \subset E_n \setminus U$ and $f \in F$ such that $y \notin f^{-1}(-\infty,1] = E_n \cap g^{-1}(-\infty,1]$, where $g \in G$ is an extension of $f$. But then $y \notin E_n \cap V \subset E_n \cap g^{-1}(-\infty,1]$, which is a contradiction. Hence $U = V \cap E_n$ and $(Q)$ holds.

**Lemma 2.** $(P) \Rightarrow E$ $\alpha$-regular.

**Proof.** Assume that $E$ is not $\alpha$-regular. Then there is a set $B$ bounded in $E$ which is not contained in any $E_n$. By taking a subsequence of $E_1, E_2, \ldots$, we may assume that for any $n \in N$ there exists $b_n \in (B \cap E_n \setminus E_{n-1}, E_0 = \{0\})$.

Since $b_1 \neq 0$, there is a closed absolutely convex neighborhood $U_1$ in $E_1$ such that $b_1 \not\subset U_1$. Also $b_2 \not\subset E_1$. Hence $\frac{1}{2}b_2 \not\subset U_1$. By $(P)$, $U_1$ is closed in $E_2$ and there exists an absolutely convex neighborhood $V_1$ in $E_2$ such that $(b_1 + V_1 + V_1) \cap U_1 = \emptyset$ and $(\frac{1}{2}b_2 + V_1 + V_1) \cap U_1 = \emptyset$. Then $U_2 = cE_2(U_1 + V_1)$ is a closed absolutely convex neighborhood in $E_2$ such that $b_1, \frac{1}{2}b_2 \not\subset U_2$. Again $U_2$ is closed in $E_3$ and $\frac{1}{2}b_3 \not\subset U_2$. Hence there is an absolutely convex neighborhood $V_2$ in $E_3$ such that $(\frac{1}{4}b_k + V_2 + V_2) \cap U_2 = \emptyset$ for $k = 1, 2, 3$. The set $U_3 = cE_3(U_2 + V_2)$ is a closed absolutely convex neighborhood in $E_3$ for which $\frac{1}{4}b_k \not\subset U_3, k = 1, 2, 3, 4$.

Once all such $U_n, n \in N$, are constructed, $\cup\{U_n; n \in N\}$ is a neighborhood in $E$ which does not absorb $B$, a contradiction.
Lemma 3. \((P) \Rightarrow E\) weakly ultraregular.

Proof. Assume \((P)\) and \(E\) not weakly ultraregular. By Lemma 2, \(E\) is \(\alpha\)-regular. Hence, there exists a set \(B\) bounded in \(E\) and \(n \in N\) such that \(B \subseteq E_n\) but \(B\) is not weakly bounded in \(E_n\). Without a loss of generality we may assume \(n = 1\).

Take a real \(f_1 \in E_1\), which is not bounded on \(B\) and choose a sequence \(b_n \in B, n \in N\), such that \(f_1(b_n) > n\). Since \((P)\) implies \((Q)\) there is a real extension \(f_2 \in E_2\) of \(f_1\) and a real extension \(f_3 \in E_3\) of \(f_2\), etc. Each set \(U_n = f_n^{-1}(\mathbb{R}, 1]\), \(n \in N\), is a closed absolutely convex neighborhood in \(E_n\) and \(U_1 \subset U_2 \subset \cdots\). Hence \(U = \bigcup\{U_n; n \in N\}\) is a \(0\)-neighborhood in \(E\). For any \(n \in N\) we have \(b_n \not\in nU\), i.e. \(U\) does not absorb \(B\) which is a contradiction.

Theorem 1. Let \((P)\) hold and each \(E_n\) be fast complete. Then \(E\) is ultraregular.

Proof. By Lemma 2, \(E\) is \(\alpha\)-regular. Let \(B \subset E\) be bounded. Then \(B \subset E_n\) for some \(n \in N\). By Lemma 3, \(B\) is weakly bounded in \(E_n\). Since \(E_n\) is fast complete, \(B\) is also bounded with respect to the topology of \(E_n\), see [4].

Lemma 4. Let each \(E_n\) be an inductive limit of metrizable spaces and \(E\) ultraregular. Then \((Q)\) holds.

Proof. Take a real \(f \in E_1, f \neq 0\). It suffices to show that \(f\) has a continuous linear extension to \(E_2\). Put \(F = (E_1, \text{top} E_2)\). Since the inclusion \(id: E_1 \to F\) is continuous, each set bounded in \(E_1\) is bounded in \(F\). On the other hand if \(B\) is bounded in \(E\) it is bounded in \(E\) and \(B \subset E_1\). Then \(B\) is bounded in \(E_1\) by the ultraregularity of \(E\). Hence the spaces \(E_1\) and \(F\) have the same families of bounded sets.

The set \(A = f^{-1}(\mathbb{R}, 1]\) absorbs all sets bounded in \(E_1\), hence it absorbs all sets bounded in \(F\). The space \(F,\) as an inductive limit of metrizable spaces, is bornological. This implies \(A\) is a \(0\)-neighborhood in \(F\). If \(a < b, x_0 \subset f^{-1}(a, b)\), and \(d = \min(f(x_0) - a, b - f(x_0))\), then \(d > 0\) and \(x_0 + dA \subset f^{-1}(a, b)\). Thus \(f^{-1}(a, b)\) is open in \(F\) and \(f^{-1}(\mathbb{R}, 1]\) is closed absolutely convex in \(E_2\) and \(f^{-1}(\mathbb{R}, 1]\) is a \(0\)-neighborhood in \(F\). The set \(M = \cap E_n, f^{-1}(\mathbb{R}, 1]\) is closed absolutely convex in \(E_2\) and \(f^{-1}(\mathbb{R}, 1]\) is a \(0\)-neighborhood in \(F\). Then \(M \cap F = M \cap E_1\).

Let \(x_1 \in E_1\) for which \(f(x_1) > 1\). Then \(x_1 \not\in M\) and there exists a real \(g \in E_2\) such that \(M \subset g^{-1}(-\mathbb{R}, 1]\) and \(g(x_1) > 1\). If \(x \not\in f^{-1}(0)\) then \(f(kx) = 0\) for each integer \(k\) which implies \(kx \in M\) and \(x \in g^{-1}(0)\). Hence \(f^{-1}(0) \subset g^{-1}(0)\) and there exists \(c > 0\) such that \(cg(x) = f(x)\) for \(x \in E_1\). The functional \(cg\) is the sought linear continuous extension of \(f\) to \(E_2\).

Theorem 2. Assume

1. Each \(E_n\) is closed in \(E_{n+1}\).
2. Each \(E_n\) is an inductive limit of metrizable spaces.
3. \(E\) is ultraregular.

Then \((P)\) holds.
Proof. By Lemma 4, assumptions 2 and 3 imply (Q) which combined with the assumption in 1 implies (P).

Theorem 3. Assume

1. Each $E_n$ is closed in $E_{n+1}$.

2. Each $E_n$ is LF-space.

Then $E$ is ultraregular iff (P) holds.

Proof. The if part follows from Theorem 2. For the only if part we observe that each LF-space $E_n$ satisfies the assumptions of the Dieudonné-Schwartz Theorem in [3]. Hence $E_n$ is ultraregular and therefore also regular. Since regular inductive limit, not necessarily strict, of Fréchet spaces is fast complete, [5], each space $E_n$ is fast complete. Then, by Theorem 1, (P) implies the ultraregularity of $E$.

References


