A DECOMPOSITION INTO HOMEOMORPHIC HANDLEBODIES WITH NATURALLY EQUIVALENT INVOLUTIONS

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ABSTRACT. Downing [6] extended the well-known result that any closed 3-manifold X contains a handlebody H such that $cl(X-H)$ is homeomorphic to H in the case where X is a compact 3-manifold with nonvoid boundary. We show that if X is a compact 3-manifold with involution h having 2-dimensional fixed point set, then X contains an h -invariant handlebody H such that the involutions induced on H and $cl(X-H)$ are naturally equivalent.

KEY WORDS AND PHRASES. 3-manifold, involution, handlebody, equivalent involutions.

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1. INTRODUCTION.

In this paper all spaces and maps are piecewise linear (PL) in the sense of Rourke and Sanderson [1]. Thus all spaces can be triangulated and all maps and subspaces are PL. This is no restriction as every 3-manifold has a triangulation by Bing [2]. Notation follows Nelson [3], [4], and Rourke and Sanderson [1].

Let X be a closed 3-manifold with triangulation T . If S is the 1-skeleton of T and H_1 is a regular neighborhood of S , then it is a classical result, see Seifert and Threlfall [5, p. 219], that H_1 and $H_2 = cl(X-H_1)$ are homeomorphic handlebodies. Downing [6] extended this result, showing that every 3-manifold with nonvoid boundary also contains a handlebody which is homeomorphic to its complement. It is the purpose of this paper to show that these results can be further extended so that certain symmetries are respected. These symmetries are those realized as involutions of X with orientable 2-dimensional fixed point sets.

Recall that an involution of X is a homeomorphism on X of period two. Involutions $h_1: X_1 \rightarrow X_1$, $i = 1, 2$, are equivalent, denoted $h_1 \sim h_2$, if there exists a homeomorphism $f: X_1 \rightarrow X_2$ such that $h_1 = f^{-1} \circ h_2 \circ f$. The fixed point set of involution $h: X \rightarrow X$ is $F(h) = \{x \in X \mid h(x) = x\}$. We shall prove the following:

THEOREM 1. Let X be an orientable 3-manifold and $h: X \rightarrow X$ an involution with 2-dimensional, orientable fixed point set F . Then there exist complementary, h -invariant handlebodies H_1 and H_2 in X such that $h|_{H_1} \sim h|_{H_2}$.

THEOREM 2. Let $f: H_1 \rightarrow H_2$ be the homeomorphism such that $h|_{H_1} = f^{-1} \circ (h|_{H_2}) \circ f$ thereby establishing the equivalence $h|_{H_1} \sim h|_{H_2}$ of Theorem 1. Then $f|_{\partial H_1}$ may be assumed to be the identity on $\partial H_1 = H_1 \cap H_2 = \partial H_2$.

By Nelson [4; Thm. C] the involutions $h|_{H_1}$ and $h|_{H_2}$ are equivalent if and only if (i) their fixed point sets are homeomorphic and (ii) either both fixed point sets separate or both fail to separate. In Nelson [3; lemma 2.1] it was shown that a 2-dimensional fixed point set of an involution on an orientable handlebody separates the handlebody if and only if the 1-dimensional fixed point set of the involution induced on the boundary separates the boundary. Since H_1 and H_2 share a common boundary, we need only show that H_1 can be chosen so that $H_1 \cap F \approx H_2 \cap F$. Hence, Theorem 1 follows immediately from the following lemma:

LEMMA 1. Let X be a compact 3-manifold and $h: X \rightarrow X$ an involution with 2-dimensional, orientable fixed point set F . Then there exist complementary, h -invariant handlebodies H_1 and H_2 in X such that $H_1 \cap F \approx H_2 \cap F$.

The 3-manifold X may be either orientable or nonorientable in this lemma. X must be assumed to be orientable in Theorem 1 in order to invoke Nelson [4; Thm. C]. Section 2 contains the proof of this lemma in the case when X is a closed manifold. The proof is modified in section 3 to cover the case when X has nonvoid boundary. The proof of Theorem 2 is provided in section 4. Both sections 3 and 4 are modifications of proofs which appear elsewhere. These sections refer heavily to the proofs which they modify.

2. X IS CLOSED.

Our approach is to pick an initial handlebody H_1 that will be repeatedly modified until $H_1 \cap F_i \approx H_2 \cap F_i$ for each component F_i of F . This identity is established by showing $H_1 \cap F_i^+ \approx H_2 \cap F_i^-$ and $H_1 \cap F_i^- \approx H_2 \cap F_i^+$, where F_i^+ and F_i^- are "homeomorphic halves" of F_i .

The Initial H_1 : We note first that any component F_i of F is a closed, orientable surface. Any such surface admits an involution g_i with a separating fixed point set consisting of at most two simple closed curves. This separates F_i into F_i^- and F_i^+ where $g_i(F_i^-) = F_i^+$.

Let X_h^* be the orbit space of h . Since F is 2-dimensional, X_h^* is a bordered 3-manifold with $\partial X_h^* \approx F$. Triangulate ∂X_h^* so that the triangulation of component F_i is g_i -invariant. Since ∂X_h^* is collared, Rourke and Sanderson [1; Cor. 2.26], the triangulation of ∂X_h^* can be extended to a triangulation T^* of X_h^* . Let T be the lift of T^* to X . Then T is an h -invariant triangulation of pair (X, F) such that F_i^+ is triangulated identically to F_i^- for each i .

Denote by $T^{(n)}$ the n th derived subdivision of T and by $S^{(n)}$ the collection of all edges and vertices of $T^{(n)}$ lying on the 1-skeleton S of T . Let $H_1 = N(S^{(2)}, T^{(2)})$ and $H_2 = (X - H_1)$. H_1 and H_2 are homeomorphic orientable handlebodies. Since T triangulates F , $H_1 \cap F_1$ is the connected, orientable surface F_1 minus k disks, where k is the number of 2-simplices in the triangulation of F_1 . $H_2 \cap F_1$ consists of k disks.

Modification Procedures: Two procedures are used to repeatedly modify T near each F_1 . The result of modification by either procedure is that the modified handlebody $\tilde{H}_1$ is isomorphic to H_1 plus a 3-handle of index one and the new complementary handlebody $\tilde{H}_2$ is isomorphic to H_2 plus a 3-handle of index one. Hence, after each modification we are again left with homeomorphic, complementary handlebodies.

The 3-simplices of T are in h -invariant pairs. We say such a pair is near F_1 if the two 3-simplices of the pair intersect in F_1 . The procedures below correspond to the cases where 3-simplices intersect in an edge or 2-face in F_1 . We will not need to employ modifications exploiting simplices intersecting F_1 in a vertex.

Procedure α : The net result of this procedure is to remove one edge from $T \cap F_1$. Intuitively, $\tilde{H}_1 \cap F_1$ contains "less" of F_1 than $H_1 \cap F_1$.

Let e be any edge of T lying in F_1 . Then e is the common edge of two 3-simplices, s_1 and s_2 , such that $h(s_1) = s_2$ and $s_1 \cap s_2 = \{e\}$. Let e_1 and e_2 be the edges of $T^{(1)}$ connecting the vertices of e with the barycenter of s_1 . Modify $S^{(2)}$ to get $\hat{S}^{(2)}$ by deleting from $S^{(2)}$ all vertices and edges of $T^{(2)}$ lying on e and adding to $S^{(2)}$ all vertices and edges of $T^{(2)}$ lying on $(e_1 \cup e_2) \cup h(e_1 \cup e_2)$. We replace H_1 by $\tilde{H}_1 = N(\hat{S}^{(2)}, T^{(2)})$. $\tilde{H}_1$ is homeomorphic to H_1 plus a 3-handle of index one. $\tilde{H}_2 = cl(X - \tilde{H}_1)$ is homeomorphic to H_2 plus a 3-handle of index one, the cocore of which lies in a disk bounded by $(e_1 \cup e_2) \cup h(e_1 \cup e_2)$.

Procedure β : The net result of this procedure is to add an h -invariant edge to T which intersects F_1 in a point. As a result, $\tilde{H}_1 \cap F_1$ contains a disk component not in $H_1 \cap F_1$.

Let $a_1 a_2 a_3$ be any 2-simplex of T in F . Then $a_1 a_2 a_3$ is the common face of 3-simplices $a_1 a_2 a_3 a_4$ and $a_1 a_2 a_3 a_5$ such that $h(a_4) = a_5$. The union of these simplices is a double tetrahedron we shall retriangulate. The new triangulation consists of the simplices $a_1 a_2 a_4 a_5$, $a_1 a_3 a_4 a_5$, $a_2 a_3 a_4 a_5$ and their faces. It is easily checked that this new triangulation is h -invariant. Let $\tilde{T}$ be the triangulation of X obtained by replacing the former simplices of the double tetrahedron by the new ones. Substitute for H_1 , $\tilde{H}_1 = |N(\tilde{S}^{(2)}, \tilde{T}^{(2)})|$. $\tilde{H}_1$ is homeomorphic to H_1 plus a 3-handle whose core is the edge $a_4 a_5$. $\tilde{H}_2 = cl(X - \tilde{H}_1)$ is also homeomorphic to H_2 plus a 3-handle of index one. **Employing the Modifications:** The α and β modifications are repeated independently near each F_1 . Let f be the number of 2-simplices and p the number of vertices in the triangulation of F_1^+ (and also of F_1^-). We assume that the original T was chosen so that $p < 2f$ for each fixed point component F_1 .

CASE 1. $p > f$. First apply β $p-f$ times where $a_1 a_2 a_3 \in F_i^-$. In order to carry this out we must have that $p-f \leq f$. But this is equivalent to the statement that $\chi(F_i^-) < 3f - e$ which is immediate when $\chi(F_i^-) = \frac{1}{2} \chi(F_i) < 0$ and easily checked in the remaining case when $F_i \approx S^2$. Next, apply procedure α to every edge in F_i^- . Then both $H_1 \cap F$ and $H_2 \cap F$ consist of p disk components and one component homeomorphic to F^+ (or F^-) minus p disks.

CASE 2. $f > p$. First apply procedure β $f-p$ times to simplices in F_i^+ . Then use procedure α to remove all edges of T from F_i^+ . The result is that both $H_1 \cap F$ and $H_2 \cap F$ consist of f disk components and one component homeomorphic to F^+ (or F^-) less f disks.

3. X IS NONVOID.

In this section we make the adjustments necessary in the above proof for the case $\partial X \neq \emptyset$. Notice that only the initial choice of H_1 was dependent upon X having empty boundary. The modification procedures do not depend upon assumptions about ∂X .

First, it was noted that F_i is a closed, orientable surface. If $\partial X \neq \emptyset$, this is no longer necessarily true. However, F_i will be a closed orientable surface less a collection of disks. Such a surface also admits an involution g_i with a separating fixed point set consisting of simple closed curves and arcs.

No longer is it true that a regular neighborhood of the 1-skeleton of triangulation of X will have a homeomorphic complement in X . Rather, we must show that Downing's construction of the handlebody H_1 can be carried out equivariantly.

Let B_i , $i = 1, \dots, n$ be the boundary components of X and let $X' = X \cup_k (UM_i)$, where M_i is the handlebody with $\partial M_i \approx B_i$ and $k|_{B_i}$ mapping B_i homeomorphically to ∂M_i . X' is a closed 3-manifold and accordingly has a classical decomposition into homeomorphic handlebodies H'_1 and $H'_2 = \text{cl}(X' - H'_1)$. By Downing [6; Lemma 1], each M_i is isotoped by G in X' onto N_i where $N_{ij} = N_i \cap H'_j$, $j = 1, 2$, are specially positioned in H_1 and H_2 . (M_i and N_i are, respectively, just regular neighborhoods of the 1-dimensional sets denoted Y and Y^1 in Downing's proof.) It is then shown that $X \approx X' - \bigcup_i M_i$ and that $H'_1 = H_1 - \bigcup_i N_{i1}$ and $H'_2 = H_2 - \bigcup_i N_{i2}$ are homeomorphic handlebodies with $H_2 = \text{cl}((X' - \bigcup_i N_i) - H_1)$.

The isotopy G of X' moving each M_i to N_i yields a homeomorphism $f: X' \rightarrow X'$ such that $f(N_i) = M_i$ for all i . $f(H_1)$ and $f(H_2)$ are homeomorphic handlebodies such that $f(H_2) = \text{cl}(X' - f(H_1))$. We do not yet know, however, that $f(H_1)$ and $f(H_2)$ are h -invariant. H'_1 and H'_2 may, however, be assumed h -invariant as in section 2. But if (i) h can be extended to an involution $h': X' \rightarrow X'$ and (ii) if the isotopy G can be constructed to commute with h' , then $f(H_1)$ and $f(H_2)$ will be h -invariant as needed.

To extend h to h' we need only know that any involution (free or with 1-dimensional fixed point set) on ∂M_i is equivalent by $k_i = k|_{B_i}$ to the restriction to B_i of an involution on M_i . Then attaching M_i to X by k_i allows h to be extended into M_i . Let $M_i = A \times I$ where A is a 2-sphere minus disks. If h is a free involution

of ∂M_i , then $h|_{M_i}$ is equivalent to ℓ given $\ell(m, t) = (\alpha(m), t)$ where α is the free involution on A derived from the antipodal map on S^2 by removing invariant pairs of disks.

If $h|_{\partial M_i}$ has a 1-dimensional fixed point set, it will consist only of simple closed curves. In this case $h|_{M_i}$ is equivalent to ℓ where α is an involution of A with arcs as fixed point components. If the fixed point set separates, then α may be derived from the reflection in a great circle of S^2 by removing invariant disks from the fixed point circle and invariant pairs of disks disjoint from the circle of fixed points.

Suppose $h|_{M_i}$ has a nonseparating fixed point set consisting of k simple closed curves. Let $M_i = A \times I$ where A is $S^1 \times S^1$ minus disks. The involution $m: S^1 \times S \rightarrow S^1 \times S^1$ given by $\gamma(s_1, s_2) = (s_2, s_1)$ has one nonseparating fixed point loop. Again, $h|_{M_i} \sim \alpha$ where α is derived from γ by removing k invariant disks from the nonseparating fixed point and an appropriate number of invariant pairs of disks disjoint from the fixed point set.

Let $A \times [-1, 1]$ be embedded in X' by j such that $h'|_j(A \times [-1, 1])$ preserves the locally defined product structure. Using an invariant collar C on B_i in X , j can be extended to an embedding, also called j , of $A \times [-1, 3]$ into X' , such that $j(A \times \{1\}) = B_i$ and so that $h'|_j(A \times [-1, 3])$ is fiber preserving. By Kim and Tollefson [7; Thm. B], $h'|_C \sim \ell$ where $\ell: B_i \times [-1, 3] \rightarrow B_i \times [-1, 3]$ is given by $\ell(b, t) = ((h'|_{B_i})(b), t)$. Since $j(A \times \{1\})$ is an invariant subspace of B_i , $j(A \times [1, 3])$ is an invariant subspace of C with $h'|_j(A \times [1, 3])$ fiber preserving. Replacing $T \times [-1, 3]$ in the proof of Downing [6; lemma 1] by $A \times [-1, 3]$, the isotopy G_t , with support in $j(A \times [-1, 3])$, taking M_i to N_i may be assumed to commute with h' . That is, we may assume that G_t preserves the fiber structure on $j(A \times [-1, 3])$.

In the case where the boundary component B_i is not h -invariant, such care in choosing the isotopy G_t is unnecessary. Any isotopy G_t , with support in a small neighborhood of M_i , will suffice as long as $h'G_t$ is applied near $M_j = hM_i$.

4. PROOF OF THEOREM 2.

In this section we show that the equivalence $h|_{H_1} \sim h|_{H_2}$ is in some sense natural. That is, we show that the homeomorphism $f: H_1 \rightarrow H_2$ establishing this equivalence may be assumed to be the identity on $\partial H_1 = H_1 \cap H_2 = \partial H_2$. This we do by modifying the proofs of Nelson [4, Theorem B] and Nelson [3, Theorem 4.1].

The proof of the equivalence in the above referenced papers is by induction on the common genus of the handlebodies H_1 and H_2 . Nelson [4; Lemma 2.1] provides $(h|_{H_i})$ -invariant, nonseparating, properly embedded disks $D_i \subset H_i$ such that cutting along the D_i lowers the genera of the fixed point sets $F(h|_{H_i})$, $i = 1, 2$. Let $H'_i = H_i - N(D_i)$, $i = 1, 2$, be the handlebodies resulting from cutting along D_i . By the induction hypothesis, $h|_{H'_1} \sim h|_{H'_2}$ and also $h|_{N(D_1)} \sim h|_{N(D_2)}$. Let these equivalences be established by $f_1: H'_1 \rightarrow H'_2$ and $f_2: N(D_1) \rightarrow N(D_2)$. By the homogeneity of

the manifolds $(H'_1)^*$ and $N(D_1)^*$, $i = 1, 2$, one may assume that $f_1(H'_1 \cap N(D_1)) = f_2(H'_2 \cap N(D_2))$. The equivalence $h|_{H_1} \sim h|_{H_2}$ is established by $f: H_1 \rightarrow H_2$ which is reconstructed from the homeomorphism f_1 and f_2 .

In order to modify the above argument so that $f|_{\partial H_1}$ is the identity on $H_1 \cap H_2$ we need only to be able to choose $D_1 \subset H_1$ and $D_2 \subset H_2$ so that $\partial D_1 = \partial D_2$. The remainder of this proof consists in showing that D_1 and D_2 can be so chosen.

Consider the disk D_1 properly embedded in H_1 and cutting $F(h|_{H_1})$ as provided by Nelson [4; Theorem B]. Let p denote the projection onto the orbit space H_1^* . Then $p(D_1)$ is a disk properly embedded in H_1^* . $\partial p(D_1) = \alpha_1 \cup \beta_1$, the union of two arcs, where $\alpha_1 \subset p(F(h|_{H_1}))$ and $\beta_1 \subset p(\partial H_1)$. D_2 can be isotoped onto D_1 relative to F if and only if α_2 can be isotoped onto α_1 relative to $p(F(h|_{H_2}))$ and β_2 can be isotoped onto β_1 relative to $p(\partial H_2)$. We can find such D_1 and D_2 by the following procedure.

Fix D_1 and regard ∂D_1 as a fixed loop on ∂H_2 . Employ Nelson [4; Theorem B] to find $D_2 \subset H_2$. If α_2 and β_2 can be isotoped relative to $p(F(h|_{H_2}))$ onto α_1 and β_1 , respectively, then we are done. If not, cut H_2 along D_2 to get a handlebody $H'_2 = \text{cl}(H_2 - N(D_2))$ of lower genus. ∂D_1 may still be regarded as a fixed loop on $\partial H'_2$. Note that with this cut the bordered surfaces $p(F \cap H'_2)$ and $p(\partial H'_2)$ which make up $\partial(H'_2)^*$ have lower genera than $p(F \cap H_2)$ and $p(\partial H_2)$, respectively. Now apply Nelson [4, Theorem B] again to H'_2 taking additional care that D'_2 misses the "scars" of previous cuts (so that D'_2 may be regarded as properly embedded in H_2). This can be done because of the homogeneity of $\partial(H'_2)^*$. Repetition of this procedure must eventually yield the desired disks D_1 and D_2 since the procedure eventually terminates with both $p(\partial H'_1)$ and $p(F \cap H'_2)$ being disks.

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