ON A THEOREM OF H. HOPF

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(Received February 1, 1989 and in revised form March 9, 1990)

ABSTRACT. A simple proof of a theorem of H. Hopf [1], via Morse theory, is given.

KEY WORDS AND PHRASES. Hypersurface, Morse function, critical point, Gauss map, degree.

1980 AMS SUBJECT CLASSIFICATION CODES. 58E05, 55.

1. INTRODUCTION AND THE THEOREM.

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a smooth map, and let

\[ V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | f(x_1, \ldots, x_n) = 0\}. \]

Suppose \( V \) is compact and the gradient, \( \nabla f \), of \( f \) is nonzero on \( V \). Then \( V \) in an \((n - 1)\)-dimensional real orientable hypersurface in \( \mathbb{R}^n \). Let \( U \) be the unbounded component of \( \mathbb{R}^n - V \). We may suppose that \( f > 0 \) on \( U \), otherwise consider \(-f\). We shall give \( V \) the following orientation. Let \( v \in V \) and let \( v_1, \ldots, v_{n-1} \) be a positively oriented basis for the tangent space \( TV_v \), regarded as a subspace of \( TR^n \). We say that \( V \) has the positive orientation at \( v \) if

\[ \text{det} \begin{bmatrix} \nabla f(v) \\ v_1 \\ \vdots \\ v_{n-1} \end{bmatrix} > 0. \]

\( V \) has the positive orientation, if it has the positive orientation at each of its points. Let \( S^{n-1} \) be the unit sphere in \( \mathbb{R}^n \), along with its usual orientation. Consider the Gauss map \( \eta : V \to S^{n-1} \) which assigns to each point of \( V \), the unit normal vector \( \nabla f/\|\nabla f\| \). Let \( d \) be the degree of \( \eta \). For a real compact manifold \( W \), let \( \chi(W) \) denote its Euler characteristic. We can now state the theorem which relates \( d \), with the Euler characteristic of certain hypersurfaces arising from \( f \).

**Theorem (Hopf [1]).** Let \( f, V, d \) be as above. Then

\[ d = \begin{cases} \frac{\chi(V)}{2} & \text{if } n \text{ is odd} \\
\chi(f \leq 0) & \text{if } n \text{ is even}. \end{cases} \]
2. Preliminaries.

The main idea of the proof of the theorem is to apply Morse theory on $V$, using a convenient Morse function. According to a theorem of Sard, the set of critical values of $\eta$ has measure zero in $S^{n-1}$ [2]. Hence, after rotating the axis if necessary, we may assume that the points $(0, \cdots, 0, \pm 1)$ are not critical values of $\eta$. Let $\pi(x_1, \cdots, x_n) = x_n$ be the projection onto the last coordinate, and let $h = \pi|_V$ be the height function on $V$. Let $p$ be a critical point of $h$. At $p$ we have:

$$\begin{align*}
f = 0, \quad \frac{\partial f}{\partial x_i} = 0, \quad i = 1, \cdots, n-1, \quad 1 = \lambda \frac{\partial f}{\partial x_n}, \quad \lambda \in \mathbb{R}.
\end{align*}$$

**Lemma 2.1** [3]. With the above considerations, $p$ is not a critical point of $\eta$, and $p$ is a nondegenerate critical point of $h$.

**Proof.** We observe that $\eta(p) = (0, \cdots, 0, \pm 1)$, since $\partial \eta_{x_n}(p) \neq 0$. Hence, $p$ is not a critical point of $\eta$. In terms of local coordinates $u_1, \cdots, u_{n-1}$ on $V$, this means that the matrix $[\frac{\partial h}{\partial u_i}, \frac{\partial h}{\partial u_j}]$, $i, j < n$, is nonsingular at $p$. In fact, near $p$ we can choose local coordinates $u_1, \cdots, u_{n-1}$ so that $x_1 = u_1, \cdots, x_{n-1} = u_{n-1}, x_n = h(u_1, \cdots, u_{n-1})$. Then,

$$\eta(u_1, \cdots, u_{n-1}) = \pm \left( \frac{\partial h}{\partial u_1}, \cdots, \frac{\partial h}{\partial u_{n-1}}, -1 \right) \sqrt{1 + \sum_{j=1}^{n-1} \left( \frac{\partial h}{\partial u_j} \right)^2}.$$

Hence, $\frac{\partial h}{\partial u_j} = \pm \frac{\partial^2 h}{\partial u_i \partial u_j}$ at $p$. Therefore, the matrix $[\frac{\partial^2 h}{\partial u_i \partial u_j}]$, $i, j < n$, is nonsingular, which implies that $p$ is a nondegenerate critical point of $h$. $\blacksquare$

Set $S = \eta^{-1}(0, \cdots, -1)$, $N = \eta^{-1}(0, \cdots, 1)$. Then the above Lemma shows that $h$ is a Morse function on $V$ with critical set $S \cup N$. For $p \in S \cup N$, we denote by $\iota(p)$ the Morse index of $h$ at $p$, which is equal to the number of negative eigenvalues, multiplicities counted, of the real symmetric matrix $[\frac{\partial^2 h}{\partial u_i \partial u_j}]$ [4].

Also, for $p \in S \cup N$ we define $\text{sgn}(p)$ to be

$$\begin{align*}
\text{sgn}(p) = \begin{cases} 
1 & \text{if near } p, \eta \text{ preserves the orientation} \\
-1 & \text{if near } p, \eta \text{ reverses the orientation.}
\end{cases}
\end{align*}$$

In addition, if $a$ is a real number, $a \neq 0$, we will denote its signature by $\text{sign}(a)$.

**Remark 2.1.** $2d = \sum_{p \in S \cup N} \text{sgn}(p)$, $\chi(V) = \sum_{p \in S \cup N} (-1)^{\iota(p)}$, [4].

We will now compute $\text{sgn}(p)$, for $p \in S \cup N$. Let $G : U \to V$ be a local parametrization of $V$ near $p$, defined by $G(x_1, \cdots, x_{n-1}) = (x_1, \cdots, x_{n-1}, h(x_1, \cdots, x_{n-1}))$. Set $\bar{p} = (p_1, \cdots, p_{n-1})$. Then,
ON A THEOREM OF H. HOPF

\[ \text{sgn } dG_p = \text{sign } \det \begin{bmatrix} \nabla f(p) \\ dG(\frac{\partial}{\partial x_1}) \\ \vdots \\ dG(\frac{\partial}{\partial x_{n-1}}) \end{bmatrix} = (-1)^{n-1} \text{sign } \frac{\partial f}{\partial x_n}(p). \]

On the other hand, if \( k : U_1 \rightarrow S^{n-1} \) is a local parametrization of \( S^{n-1} \) near the point \((0, \cdots, 0, \frac{\partial f}{\partial x_n}(p)/|\frac{\partial f}{\partial x_n}(p)|)\), defined by \( k(s_1, \cdots, s_{n-1}) = (s_1, \cdots, s_{n-1}, \text{sign } \frac{\partial f}{\partial x_n}(p)\sqrt{1 - \sum s_i^2}) \), then,

\[ \text{sgn } dk_0 = \text{sign } \det \begin{bmatrix} \nabla f(p) \\ dk(\frac{\partial}{\partial s_1}) \\ \vdots \\ dk(\frac{\partial}{\partial s_{n-1}}) \end{bmatrix} = (-1)^{n-1} \text{sign } \frac{\partial f}{\partial x_n}(p). \]

Also, near \( p \), \( \eta = -\text{sign } \frac{\partial f}{\partial x_n}(p)\frac{\nabla h, -1}{\sqrt{1 + \sum (\frac{\partial h}{\partial u_i})^2}}. \) Hence,

\[ \text{sgn}(p) = \text{sgn } d(k^{-1} \circ \eta \circ G)(p) = \left(-\text{sign } \frac{\partial f}{\partial x_n}(p)\right)^{n-1} \cdot \text{sign } \det \left[ \frac{\partial^2 h}{\partial u_i \partial u_j} \right]. \tag{2.1} \]

**Lemma 2.2.** For \( p \in S \cup N \), \( \text{sgn}(p) = -\text{sign } \det BH(f)(p) \), where \( BH(f) = \begin{bmatrix} 0 & \nabla(f) \\ \nabla^T f & H(f) \end{bmatrix} \) is the Bordered Hessian matrix of \( f \).

**Proof.** We have \( f(u_1, \cdots, u_{n-1}, h(u_1, \cdots, u_{n-1})) = 0 \), where \( u_1, \cdots, u_{n-1}, h \), are as in Lemma 2.1. By differentiating the above identity twice, and evaluating at \( p \), we get

\[ \frac{\partial^2 f}{\partial x_i \partial x_j} + \frac{\partial f}{\partial x_n}(p)\frac{\partial^2 h}{\partial u_i \partial u_j} = 0, \quad 1 \leq i, j \leq n - 1. \tag{2.2} \]

Using (2.1) we get \( \text{sgn}(p) = \left(-\text{sign } \frac{\partial f}{\partial x_n}(p)\right)^{n-1} \cdot \text{sign } \det \left[ \frac{\partial^2 h}{\partial u_i \partial u_j} \right] = \text{sign } \det BH(f)(p) \).

**Remark 2.2.** If \( n \) is even, then \( \chi(V) = 0. \)

**Proof.** We have \( \chi(V) = \sum_{p \in S} (-1)^i(p) + \sum_{p \in N} (-1)^i(p). \) But if \( p \in S \) then \( \text{sgn}(p) = (-1)^i(p) \), while if \( p \in N \), \( \text{sgn}(p) = (-1)(-1)^i(p). \) Hence,

\[ \chi(V) = \sum_{p \in S} \text{sgn}(p) - \sum_{p \in N} \text{sgn}(p) = d - d = 0. \]
3. Proof of the Theorem. Case i. n is odd. We observe from (2. 1), that
\[ sgn(p) = \text{sign} \det \left[ \frac{\partial^2 V}{\partial x_i \partial x_j} \right] = (-1)^{t(p)}. \] Hence, by Remark 2. 1,
\[ \chi(V) = \sum_{p \in S \cup N} (-1)^{t(p)} = \sum_{p \in S \cup N} sgn(p) = 2d. \]

Case ii. n is even. Then, let us consider \( V^- = \{ f \leq 0 \} \). This is a compact orientable manifold with boundary \( V \). Consider the double covering \( W \) of \( V^- \), ramified along \( V \), which is defined by
\[ W = \{(x_1, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1} | f(x_1, \ldots, x_n) + x_{n+1}^2 = 0 \}. \]
\( W \) is a compact \( n \)-dimensional nonsingular hypersurface and \( \chi(W) = 2\chi(V^-) - \chi(V) = 2\chi(V^-) \), since \( n \) is even. We orient \( W \) as we oriented \( V \). On \( W \) we consider the height function \( \tilde{h} \) where \( \tilde{h} = \pi|_W, \pi(x_1, \ldots, x_n, x_{n+1}) = x_n \). Let \( \tilde{\eta} : W \to S^n \) be the Gauss map, and let \( \tilde{d} \) be its degree. Regard \( \mathbb{R}^n, S, N \) as subsets of \( \mathbb{R}^{n+1} \).

As in Lemma 2.1, we have that if \( p \in S \cup N \), then \( p \) is a nondegenerate critical point of \( \tilde{h} \). In fact, \( S \cup N \) is the critical set of \( \tilde{h} \), and the points \((0, \ldots, 0, \pm 1, 0)\) are not critical values of \( \tilde{h} \). Let now \( p \in S \cup N \). \( p \) is viewed as a critical point of both \( \tilde{h} \) and \( \tilde{h} \), and also as a noncritical point of \( \tilde{h} \) and \( \tilde{h} \). Denote by \( \tilde{sgn}(p) \), the \( sgn(p) \) viewed as a noncritical point of \( \tilde{h} \). We have:
\[ sgn(p) = (-1)\text{sign} \det \left[ \begin{array}{cc} 0 & \nabla f \\ \nabla f & H(f) \end{array} \right] = (-1)\text{sign} \det \left[ \begin{array}{ccc} 0 & \nabla f & 0 \\ \nabla f & H(f) & 0 \\ 0 & 0 & 2 \end{array} \right] = \tilde{sgn}(p). \]

Hence, \( d = \tilde{d} = \chi(W) = \chi(V^-) = \chi(f \leq 0) \). The proof of the theorem is now complete. \( \blacksquare \)

References
