COMMON FIXED POINTS OF COMPATIBLE MAPPINGS

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ABSTRACT. In this paper, we present a common fixed point theorem for compatible mappings, which extends the results of Ding, Diviccaro-Sessa and the third author.

KEY WORDS AND PHRASES. Common fixed points, commuting mappings, weakly commuting mappings and compatible mappings.

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1. INTRODUCTION.

In [1], the concept of compatible mappings was introduced as a generalization of commuting mappings. The utility of compatibility in the context of fixed point theory was demonstrated by extending a theorem of Park-Bae [2]. In [3], the third author extended a result of Singh-Singh [4] by employing compatible mappings in lieu of commuting mappings and by using four functions as opposed to three. On the other hand, Diviccaro-Sessa [5] proved a common fixed point theorem for four mappings, using a well known contractive condition of Meade-Singh [6] and the concept of weak commutativity of Sessa [7]. Their theorems generalize results of Chang [8], Imdad Khan [9], Meade-Singh [6], Sessa-Fisher [10] and Singh-Singh [4].

In this paper, we extend the results of Ding [11], Diviccaro-Sessa [5] and the third author [3].

The following Definition 1.1 is given in [1].

DEFINITION 1.1. Let A and B be mappings from a metric space $(X,d)$ into itself. Then A and B are said to be compatible if $\lim_{n \to \infty} d(ABx_n, BAx_n) = 0$ whenever $(x_n)$ is a sequence in $X$ such that $\lim_{n \to \infty} A^{n}x = z$ and $\lim_{n \to \infty} B^{n}x = z$ for some $z$ in $X$.

Thus, if $d(ABx_n, BAx_n) + 0$ as $d(Ax_n, Bx_n) + 0$, then A and B are compatible.
Mappings which commute are clearly compatible, but the converse is false. S. Sessa [7] generalized commuting mappings by calling mappings $A$ and $B$ from a metric space $(X,d)$ into itself a weakly commuting pair if $d(ABx, BAx) \leq d(Ax, Bx)$ for all $x$ in $X$. Any weakly commuting pair are obviously compatible, but the converse is false [3]. See [1] for other examples of the compatible pairs which are not weakly commutative and hence not commuting pairs.

**Lemma 1.1 ([1]).** Let $A$ and $B$ be compatible mappings from a metric space $(X,d)$ into itself. Suppose that $\lim_{n \to \infty} A_{x_n} = \lim_{n \to \infty} B_{x_n} = z$ for some $z$ in $X$. Then $\lim_{n \to \infty} BA_{x_n} = A_z$ if $A$ is continuous.

2. A FIXED POINT THEOREM.

Throughout this paper, suppose that the function $\Psi : [0, \infty)^5 \to [0, \infty)$ satisfies the following conditions:

1. $\Psi$ is nondecreasing and upper semicontinuous in each coordinate variable,
2. For each $t > 0$, $\Psi(t) = \max \{\Psi(0,0,t,t,t), \Psi(t,t,t,2t,0), \Psi(t,t,t,0,2t)\} < t$. (2.1)

**Lemma 2.1 ([12]).** Suppose that $\Psi : [0, \infty) \to [0, \infty)$ is nondecreasing and upper semicontinuous from the right. If $\Psi(t) < t$ for every $t > 0$, then $\lim_{n \to \infty} \Psi(t) = 0$, where $\Psi^n(t)$ denotes the composition of $\Psi(t)$ with itself $n$-times.

Now, we are ready to state our main Theorem.

**Theorem 2.2.** Let $A, B, S$, and $T$ be mappings from a complete metric space $(X,d)$ into itself. Suppose that one of $A, B, S$, and $T$ is continuous, the pairs $A, S$ and $B, T$ are compatible and that $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. If the inequality

$$d(Ax, By) \leq \Psi(d(Ax, Sx), d(By, Ty), d(Sx, Ty), d(Ax, Ty), d(By, Sx))$$

holds for all $x$ and $y$ in $X$, where $\Psi$ satisfies (1) and (2), then $A, B, S$, and $T$ have a unique common fixed point in $X$.

**Proof.** Let $x_0 \in X$ be given. Since $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$, we can choose $x_1$ in $X$ such that $y_1 = T_{x_1} = A_{x_0}$ and, for this point $x_1$, there exists a point $x_2$ in $X$ such that $y_2 = S_{x_2} = B_{x_1}$ and so on. Inductively, we can define a sequence $\{y_n\}$ in $X$ such that

$$y_{n+1} = T_{y_n} = A_{x_n} \quad \text{and} \quad y_{n+2} = S_{y_n} = B_{x_{n+1}}.$$ (2.3)

By (2.2) and (2.3), we have

$$d(y_{n+1}, y_{n+2}) = d(A_{x_n}, B_{x_{n+1}})$$

$$< \Psi(d(A_{x_n}, S_{x_n}), d(B_{x_{n+1}}, T_{x_{n+1}}), d(S_{x_n}, T_{x_{n+1}}),$$

$$d(A_{x_n}, T_{x_{n+1}}), d(B_{x_{n+1}}, S_{x_n}))$$

$$< \Psi(y_{n+1}, y_n), d(y_{n+2}, y_{n+1}), d(y_n, y_{n+1}),$$

$$0, d(y_{n+2}, y_n))$$

$$< \Psi(y_n, y_{n+1}, d(y_{n+1}, y_{n+2}), d(y_n, y_{n+1}),$$

$$0, d(y_{n+1}, y_{n+2})).$$
If \( d(y_{2n+1}, y_{2n+2}) > d(y_{2n}, y_{2n+1}) \) in the above inequality, then we have
\[
d(y_{2n+1}, y_{2n+2}) \leq \phi(d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}),
\]
\[
d(y_{2n+1}, y_{2n+2}), 0, 2d(y_{2n+1}, y_{2n+2}))
\]
\[
\leq \phi(d(y_{2n+1}, y_{2n+2})) < d(y_{2n+1}, y_{2n+2}),
\]
which is a contradiction. Thus,
\[
d(y_{2n+1}, y_{2n+2}) < \phi(d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}),
\]
\[
0, 2d(y_{2n}, y_{2n+1}))
\]
\[
\phi(d(y_{2n}, y_{2n+1})).
\]
Similarly, we have
\[
d(y_{2n+2}, y_{2n+3}) < \phi(d(y_{2n+1}, y_{2n+2})).
\]
(2.5)
It follows from (2.4) and (2.5) that
\[
d_n = d(y_n, y_{n+1}) < \phi(d(y_{n-1}, y_n)) < \ldots < \phi^{-1}(d(y_1, y_2)).
\]
(2.6)
By (2.6) and Lemma 2.1, we obtain
\[
\lim_{n \to \infty} d_n = 0.
\]
(2.7)
In order to show that \( \{y_n\} \) is a Cauchy sequence, it is sufficient to show
that \( \{y_{2n}\} \) is a Cauchy sequence. Suppose that \( \{y_{2n}\} \) is not a Cauchy sequence. Then
there is an \( \varepsilon > 0 \) such that, for each even integer \( 2k \), there exist even integers \( 2m(k) \)
and \( 2n(k) \) such that
\[
d(y_{2m(k)}, y_{2n(k)}) > \varepsilon \text{ for } 2m(k) > 2n(k) > 2k.
\]
(2.8)
For each even integer \( 2k \), let \( 2m(k) \) be the least even integer exceeding
\( 2n(k) \) satisfying (2.8), that is,
\[
d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon \text{ and } d(y_{2n(k)}, y_{2m(k)}) > \varepsilon.
\]
(2.9)
Then, for each even integer \( 2k \),
\[
\varepsilon < d(y_{2n(k)}, y_{2m(k)}) < d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}.
\]
It follows from (2.7) and (2.9) that
\[
\lim_{k \to \infty} d(y_{2n(k)}, y_{2m(k)}) = \varepsilon.
\]
(2.10)
By the triangle inequality,
\[
|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1} \text{ and}
\]
\[
|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1} + d_{2n(k)}.
\]
From (2.7) and (2.10), as \( k \to \infty \),
\[
d(y_{2n(k)}, y_{2m(k)-1}) < \varepsilon \text{ and } d(y_{2n(k)+1}, y_{2m(k)-1}) < \varepsilon.
\]
By (2.2) and (2.3), we have
\[ d(Y_{2n}(k), Y_{2m}(k)) \leq d(Y_{2n}(k), Y_{2m}(k)) + d(A_{2n}, B_{2m}) + d(X_{2n}, S_{2m}) + d(Y_{2n}(k), Y_{2m}(k)). \]

Since \( \Phi \) is upper semicontinuous,
\[ \varepsilon < \Phi(0, 0, \varepsilon, \varepsilon) < \varepsilon \text{ as } k \to \infty, \]
which is a contradiction. Hence \( \{y_n\} \) is a Cauchy sequence and it converges to some point \( z \) in \( X \). Consequently the subsequences \( \{A_{2n}\}, \{S_{2n}\}, \{B_{2n}\} \) and \( \{T_{2n}\} \) converge to \( z \). Suppose that \( S \) is continuous. Since \( A \) and \( S \) are compatible, Lemma 1.2 implies that
\[ SS_{2n} \text{ and } AS_{2n} \to Sz. \]
By (2.2), we obtain
\[ d(SS_{2n}, TX_{2n-1}) \leq \Phi(d(A_{2n}, SS_{2n}), d(B_{2n-1}, TX_{2n-1}), d(SS_{2n}, TX_{2n-1}), d(AS_{2n}, TX_{2n-1}), d(B_{2n-1}, SS_{2n})). \]

Letting \( n \to \infty \), we have
\[ d(Sz, z) \leq \Phi(0, 0, d(Sz, z), d(z, Sz)), \]
so that \( z = Sz \). By (2.2), we also obtain
\[ d(Az, Bx_{2n-1}) \leq \Phi(d(Az, Sz), d(Bx_{2n-1}, Tx_{2n-1}), d(Sz, Tx_{2n-1}), d(Az, Tx_{2n-1}), d(Bx_{2n-1}, Sz)). \]

Letting \( n \to \infty \), we have
\[ d(Az, z) \leq \Phi(0, 0, d(Az, z), d(z, Az)), \]
so that \( z = Az \). Since \( A(X) \subseteq T(X) \), \( z \in T(X) \) and hence there exists a point \( w \) in \( X \) such that \( z = Aw = Tw \).
\[ \Phi(d(Az, Bw), d(Bw, Tw), d(Sz, Tw), d(Az, Tw), d(Bw, z)) \]
which implies that \( z = Bw \). Since \( B \) and \( T \) are compatible and \( Tw = Bw = z \), \( d(Tw, BTw) = 0 \) and hence \( Tz = TBw = BTw = Bz \). Moreover, by (2.2),
\[ d(z, Tz) = d(Az, Bz) \leq \Phi(0, d(Bz, Tz), d(z, Tz), d(z, Tz), d(Bz, z)), \]
so that \( z = Tz \). Therefore, \( z \) is a common fixed point of \( A, B, S \) and \( T \). Similarly, we can complete the proof in the case of the continuity of \( T \). Now, suppose that \( A \) is continuous. Since \( A \) and \( S \) are compatible, Lemma 1.2 implies that
\[ AA_{2n} \text{ and } SA_{2n} \to Az. \]
By (2.2), we have
\[ d(AA_{2n}, Bx_{2n-1}) \leq \Phi(d(AA_{2n}, SA_{2n}), d(Bx_{2n-1}, TX_{2n-1}), d(SA_{2n}, TX_{2n-1}), d(AA_{2n}, TX_{2n-1}), d(Bx_{2n-1}, SA_{2n})). \]

Letting \( n \to \infty \), we obtain
\[ d(Az, z) \leq \Phi(0, 0, d(Az, z), d(z, Az)), \]
so that \( z = Az \). Hence, there exists a point \( v \) in \( X \) such that \( z = Az = Tv \).
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\[ d(\text{AA}x_{2n}, Bv) \leq \Phi(d(\text{AA}x_{2n}, \text{SA}x_{2n}), d(Bv, Tv), d(\text{SA}x_{2n}, Tv)), \]
\[ d(\text{AA}x_{2n}, Tv) d(Bv, \text{SA}x_{2n})) , \]

Letting \( n \to \infty \), we have

\[ d(z, Bv) < \Phi(0, d(Bv, Tv), d(z, Tv), d(Az, Tv), d(Bv, z)), \]

which implies that \( z = Bv \). Since \( B \) and \( T \) are compatible and \( Tv = Bv = z \), \( d(TBv, BTv) = 0 \) and hence \( Tz = TBv = BTv = Bz \). Moreover, by (2.2), we have

\[ d(\text{AA}x_{2n}, Bz) < \Phi(d(\text{AA}x_{2n}, \text{SA}x_{2n}), d(Bz, Tz), d(\text{SA}x_{2n}, Tz)), \]
\[ d(\text{AA}x_{2n}, Tz), d(Bz, \text{SA}x_{2n})). \]

Letting \( n \to \infty \), \( d(z, Bz) < \Phi(0, d(Bz, Tz), d(z, Tz), d(z, Tz), d(Bz, z)), \) so that \( z = Bz \). Since \( B(X) \subseteq S(X) \), there exists a point \( w \) in \( X \) such that \( z = Bz = Sw \).

\[ d(Aw, z) = d(Aw, Bz) < \Phi(d(Aw, Sw), 0, d(Sw, z), d(Aw, z), d(z, Sw)), \]

so that \( Aw = z \). Since \( A \) and \( S \) are compatible and \( Aw = Sw = z \), \( d(SBw, BSw) = 0 \) and hence \( Sz = SAw = ASw = Az \). Therefore \( z \) is a common fixed point of \( A, B, S \) and \( T \). Similarly, we can complete the proof in the case of the continuity of \( B \). It follows easily from (2.2) that \( z \) is a unique common fixed point of \( A, B, S \) and \( T \).

COROLLARY 2.3. Let \( A, B, S \) and \( T \) be mappings from a complete metric space \((X,d)\) into itself. Suppose that one of \( A, B, S \) and \( T \) is continuous, the pairs \( A,S \) and \( B,T \) are compatible and that \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \). If the inequality (2.2) holds for all \( x \) and \( y \) in \( X \), where \( \Phi \) satisfies (1) and (2.11);

\[ \Phi(t) = \max\{\Phi(t,t,t,t), \Phi(t,t,t,2t), \Phi(t,t,t,0,2t)} < t \quad (2.11) \]

for each \( t > 0 \), then \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

REMARK 2.4. From Theorem 2.2 and Corollary 2.3, we extend the results of Ding [11] and Diviccaro-Sessa [5] by employing compatibility in lieu of commuting and weakly commuting mappings, respectively. Further our theorem extends also a result of Ding [11] by using one continuous function as opposed to two.

REMARK 2.5. From Theorem 2.2 defining \( \Phi: [0, \infty) \to [0, \infty) \) by

\[ \Phi(t_1, t_2, t_3, t_4, t_5) = \max\{t_1, t_2, t_3, t_4, t_5\} \]
\[ \max\{t_1, t_2, t_3, t_4, t_5\} \}

for all \( t_1, t_2, t_3, t_4, t_5 \in [0, \infty) \) and \( h \in [0, 1) \), we obtain a result of the third author [3] even if one function is continuous as opposed to two.

REFERENCES


5. DIVICCARO, M.L. and SESSA, S., Some remarks on common fixed points of four mappings, Jnanabha, 15 (1985), 139-149.


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