ON PROBABILISTIC NORMED SPACES UNDER $\tau_{T,L}$

MINGSHENG YING

Department of Mathematics
Fuzhou Teacher's College
Jiangxi, China

(Received June 20, 1988 and in revised form September 12, 1988)

ABSTRACT. We introduce the operation $\oplus_L$ copulative with $\tau_{T,L}$ to define PN space under $\tau_{T,L}$ and establish some basic properties of probabilistic seminorms and norms under $\tau_{T,L}$. Finally, we discuss so-called L-simple spaces.

KEY WORDS AND PHRASES. Probabilistic normed space, L-simple space.

1980 AMS SUBJECT CLASSIFICATION CODES. Primary 46B99.

1. INTRODUCTION.

In [1-4], Šerstnev introduced the concept of PN space. A triple (V, ν, τ) is called a PN space, if V is a vector space over the field K of real or complex numbers, ν is a function from V into Δ^+ , the set of all distance distribution functions, τ is a continuous triangle function, and for any $p, q \in V$, $a \in K$ with $a \neq 0$, the following conditions hold.

$$(i) \nu(0) = \varepsilon_0, \quad (1.1)$$

$$(ii) \nu(p) \neq \varepsilon_0 \text{ if } p \neq 0, \quad (1.2)$$

$$(iii) \nu(ap) = |a| \odot \nu(p), \quad (1.3)$$

$$(iv) \nu(p+q) \geq \tau(\nu(p), \nu(q)), \quad (1.4)$$

where $|a| \odot \nu(p) = \nu(p)(j/|a|)_\tau$ and j denotes the identity function. Since $\odot$ and τ are not always cooperative as multiplication and addition, there is a certain difficulty in the further development of PN space theory. In fact, for any $p, q \in V$, $a \in K$ with $a > 0$, we can estimate $\nu(ap+aq)$ in two ways and the two estimates are not always consistent (see Schweizer and Sklar [5], p 238). To overcome this objection, Muštari and Šerstnev [6-7] had to focus their attention on homogeneous triangle functions.

In this paper, we establish the operation $\oplus_L$ copulative with $\tau_{T,L}$ and use it to discuss PN spaces under $\tau_{T,L}$, where $\tau_{T,L}: \Delta^+ \times \Delta^+ \rightarrow \Delta^+$,

$\tau_{T,L}(F,G)(x) = \sup \{T(F(u), G(v)) \mid L(u,v) = x\}$, $x \in R^+$, $F, G \in \Delta^+$, T is a continuous t -norm, and $L: R^+ \times R^+ \rightarrow R^+$ satisfies:

- 1) $\text{Ran} L = R$;
- 2) L has 0 as identity;
- 3) L is a nondecreasing in each place, and if $u_1 < u_2$, $v_1 < v_2$,

then $L(u_1, v_1) < L(u_2, v_2)$;

4) L is continuous on $R^+ \times R^+$, except possibly at the points $(0, \infty)$ and $(\infty, 0)$;

5) L is associative;

6) L is Archimedean, i.e., for all $u \in (0, \infty)$, $L(u, u) > u$.

First, we give some simple results which are needed in the sequel. From Theorem 5.7.4 in [5], it is easy to know that there exists an additive generator g of L , i.e., a strictly increasing and continuous function $g: R^+ \rightarrow R^+$ with $g(0) = 0$, $g(\infty) = \infty$, such that $L(x, y) = g^{-1}(g(x) + g(y))$, $x, y \in R^+$.

Now, we choose a fixed additive generator g of L , and note that the particular choice of g does not affect the validity of our results.

DEFINITION 1.1. $\alpha_L^*: R^+ \times R^+ \rightarrow R^+$ is defined as

$$\alpha_L^* x = g^{-1}(g(\alpha x)), \alpha, x \in R^+.$$

Clearly, $\alpha_{\text{sum}}^* x = \alpha x$, $\alpha, x \in R^+$.

LEMMA 1.1. For any $\alpha, \beta, x, y \in R^+$, the following equalities hold.

$$(i) \quad \alpha_L^* (\beta_L^* x) = (\alpha\beta)_L^* x \quad (1.5)$$

$$(ii) \quad \alpha_L^* L(x, y) = L(\alpha_L^* x, \alpha_L^* y) \quad (1.6)$$

$$(iii) \quad (\alpha\beta)_L^* x = L(\alpha_L^* x, \beta_L^* x) \quad (1.7)$$

Clearly, if $\alpha \in (0, \infty)$, then $f(x) = \alpha_L^* x$, $x \in R^+$ is strictly increasing and continuous.

So we may give

DEFINITION 1.2. For any $\alpha \in (0, \infty)$, $x \in R^+$, $x \delta_L \alpha$ is defined as the only solution of the equation $\alpha_L^* t = x$.

LEMMA 1.2. For any $\alpha, \beta \in (0, \infty)$, $x, y \in R^+$, the following equalities hold.

$$(i) \quad (x \delta_L \alpha) \delta_L \beta = x \delta_L (\alpha\beta), \quad (1.8)$$

$$(ii) \quad L(x, y) \delta_L \alpha = L(x \delta_L \alpha, y \delta_L \alpha). \quad (1.9)$$

DEFINITION 1.3. $\odot_L: (0, \infty) \times \Delta^+ \rightarrow \Delta^+$ is defined as

$$\odot_L F = F(j \delta_L \alpha), \alpha \in (0, \infty), F \in \Delta^+.$$

In particular, $\odot_{\text{sum}} F = \alpha \odot F$, $\alpha \in (0, \infty)$, $F \in \Delta^+$.

LEMMA 1.3. For any $\alpha, \beta \in (0, \infty)$, $x \in R^+$, $F, G \in \Delta^+$, the following equalities hold.

$$(i) \quad \odot_L \varepsilon_x = \varepsilon_{\alpha_L^* x}, \quad (1.10)$$

$$(ii) \quad \odot_L (\odot_L F) = (\alpha\beta)_L \odot_L F, \quad (1.11)$$

$$(iii) \quad \odot_{T, L}(F, G) = \tau_{T, L}(\odot_L F, \odot_L G). \quad (1.12)$$

COROLLARY 1.1. (cf. Lemma 15.1.3 in [5]) For any $\alpha \in (0, \infty)$, $F, G \in \Delta^+$,

$$\tau_{T, L}(F, G) = \tau_{T, L}(F(a_L^* j), G(a_L^* j))(j \delta_L a), \quad (1.13)$$

i.e., $\tau_{T, L}$ is homogenous in the sense of (1.13).

DEFINITION 1.4. For any $x, y \in [0, \infty)$ with $y < x$, $x_L \sim y$ is defined as the only solution of the equation $L(y, t) = x$.

DEFINITION 1.5. For any $a, b \in I$,

$$a \alpha_T b = \text{Sup}\{x | T(b, x) < a\}.$$

LEMMA 1.4. For any $a, b, c, a_\lambda, b_\lambda (\lambda \in \Lambda) \in I$,

$$(i) \quad T(a, b)_{\alpha_T} a > b, \quad (1.14)$$

$$(ii) \quad \text{If } a < b, \text{ then } a_{\alpha_T c} < b_{\alpha_T c}, c_{\alpha_T b} \leq c_{\alpha_T a}, \quad (1.15)$$

$$(iii) \quad a_{\alpha_T} \inf_{\lambda \in \Lambda} b_\lambda > \sup_{\lambda \in \Lambda} (a_{\alpha_T} b_\lambda), \quad (1.16)$$

$$a_{\alpha_T} \sup_{\lambda \in \Lambda} b_\lambda = \inf_{\lambda \in \Lambda} (a_{\alpha_T} b_\lambda), \quad (1.17)$$

$$\inf_{\lambda \in \Lambda} a_\lambda \alpha_T b = \inf_{\lambda \in \Lambda} (a_\lambda \alpha_T b), \quad (1.18)$$

$$\sup_{\lambda \in \Lambda} a_\lambda \alpha_T b > \sup_{\lambda \in \Lambda} (a_\lambda \alpha_T b). \quad (1.19)$$

DEFINITION 1.6. $\eta_{T,L}: \Delta^+ \times \Delta^+ \rightarrow I^{R^+}$ is defined as: for all $F, G \in G$,

$$(F \eta_{T,L} G)(x) = \begin{cases} \inf \{F(u)_{\alpha_T} G(v) \mid u_L v = x, v < u, u, v \in [0, \infty)\}, & x \in [0, \infty); \\ 1, & x = \infty. \end{cases}$$

It is easy to check that for any $F, G \in \Delta^+$, $F \eta_{T,L} G$ is left-continuous and increasing, but it is possible that $(F \eta_{T,L} G)(0) > 0$. In addition, from Lemma 2.4. (ii), we know that $\eta_{T,L}$ is increasing in the first place and decreasing in the second place.

LEMMA 1.5. For any $F, G \in \Delta^+$,

$$\tau_{T,L}(F, G) \eta_{T,L} G > F. \quad (1.20)$$

2. PROBABILISTIC SEMINORMS AND NORMS UNDER $\tau_{T,L}$

DEFINITION 2.1. Let V be a vector space over the field K of real or complex numbers, $v: V \rightarrow \Delta^+$. Then (V, v) is called a PSN space under $\tau_{T,L}$ if for all $p, q \in V$, $\alpha \in K$ with $\alpha \neq 0$, the following conditions hold.

$$(i) \quad v(0) = \varepsilon_0, \quad (2.1)$$

$$(ii) \quad v(\alpha p) = |\alpha| \otimes_L v(p), \quad (2.2)$$

$$(iii) \quad v(p+q) > \tau_{T,L}(v(p), v(q)). \quad (2.3)$$

If (V, v) is a PSN space and satisfies: for all $p \in V$,

$$(iv) \quad v(p) \neq \varepsilon_0 \text{ if } p \neq 0, \quad (2.4)$$

then (V, v) is called a PN space.

THEOREM 2.1. If (V, v) is a PSN space under $\tau_{T,L}$, then for all $p, q \in V$,

$$v(p-q) < M(v(p) \eta_{T,L} v(q), v(q) \eta_{T,L} v(p)), \quad (2.5)$$

where M denotes the minimum function.

PROOF. From Lemma 1.5, we have

$$v(p) \eta_{T,L} v(q) > \tau_{T,L}(v(p-q), v(q)) \eta_{T,L} v(q) > v(p-q)$$

because $v(p) > \tau_{T,L}(v(p-q), v(q))$.

In addition,

$$v(p-q) = 1 \otimes_L v(q-p)$$

$$= v(q-p)$$

$$< v(q) \eta_{T,L} v(p).$$

THEOREM 2.2. In a PSN space (V, v) under $\tau_{T,L}$, for all $\varepsilon \in R^+$, $\lambda \in I$, $p \in V$, the ball with center p and radius ε of level λ $B_p(\varepsilon, \lambda) = \{q \mid T(v_{q-p}(\varepsilon), \lambda) = \lambda\}$ is convex.

PROOF. If $q_1, q_2 \in B_p(q, \lambda)$, $t \in [0, 1]$, then

$$\begin{aligned}
 & v_{[tq_1 + (1-t)q_2] - p}(\varepsilon) = v_{t(q_1 - p) + (1-t)(q_2 - p)}(\varepsilon) \\
 & > \tau_{T, L}(v_{t(q-p)}, v_{(1-t)(q-p)})(\varepsilon) \\
 & = \sup \{T(v_{t(q_1 - p)}(\varepsilon_1), v_{(1-t)(q_2 - p)}(\varepsilon_2)) \mid L(\varepsilon_1, \varepsilon_2) = \varepsilon\} \\
 & > T(v_{t(q_1 - p)}(t^*_{L, \varepsilon}), v_{(1-t)(q_2 - p)}((1-t)^*_{L, \varepsilon})) \\
 & = T(v_{q_1 - p}(\varepsilon), v_{q_2 - p}(\varepsilon)). \\
 & T(v_{[tq_1 + (1-t)q_2] - p}(\varepsilon), \lambda) > T(T(v_{q_1 - p}(\varepsilon), v_{q_2 - p}(\varepsilon)), \lambda) \\
 & = T(v_{q_1 - p}(\varepsilon), T(v_{q_2 - p}(\varepsilon), \lambda)) \\
 & = T(v_{q_1 - p}(\varepsilon), \lambda) \\
 & = \lambda.
 \end{aligned}$$

THEOREM 2.3. In a PN space (V, v) under $\tau_{T, L}$, let

$$U_p(\varepsilon, \lambda) = \{q \mid v_{q-p}(\varepsilon) > 1-\lambda\}, \quad \varepsilon, \lambda > 0, p \in V.$$

Then the family $\mathcal{U} = \{U_p(\varepsilon, \lambda) \mid \varepsilon > 0, \lambda > 0, p \in V\}$ generates a Hausdorff topology τ which is called the strong topology of V . Moreover,

- (1) $+$: $V \times V \rightarrow V$, $(p, q) \mapsto p+q$, $p, q \in V$ is continuous;
- (2) If $\text{Ran } v \subseteq \mathcal{D}^+$, then: $k \times V \rightarrow V$, $(\alpha, p) \mapsto \alpha p$, $\alpha \in R$, $p \in V$ is continuous, where $\mathcal{D}^+ = \{F \in \Delta^+ \mid \sup_{x < \infty} F(x) = 1\}$;
- (3) v : $V \rightarrow \Delta^+$, $p \mapsto v(p)$, $p \in V$ is continuous.

PROOF. Straightforward.

To illustrate that the condition in Theorem 2.3. (2) is necessary, we give

EXAMPLE 2.2. Let $v_0: R \rightarrow \Delta^+$

$$v_0(x) = \begin{cases} \varepsilon_0, & \text{if } x = 0, \\ \varepsilon_\infty, & \text{if } x \neq 0. \end{cases}$$

Then (R, v_0) is a PN space under $\tau_{T, L}$. However, $1/n \rightarrow 0$, but $1/n \xrightarrow{\mathcal{P}(R, v_0)} 0$ does not hold.

THEOREM 2.4. If (V, v) is a PSN (or PN) space under $\tau_{T, L}$, $\mathcal{F}: V \times V \rightarrow \Delta^+$ is defined as

$$F(p, q) = v(p-q), \quad p, q \in V, \quad (2.6)$$

then (V, F) is a PPM (resp. PM) space under $\tau_{T, L}$ which has the following properties: for all $p, q, r \in V$, $\alpha \in K$ with $\alpha \neq 0$,

$$(i) \quad F(\alpha p, \alpha q) = |\alpha| \otimes_L F(p, q), \quad (2.7)$$

$$(ii) \quad F(p+r, q+1) = F(p, q). \quad (2.8)$$

Conversely, if (V, F) is a PPM (or PM) space under $\tau_{T, L}$ with (2.7), (2.8), then there exists a PSN (resp. PN) space under $\tau_{T, L}$ such that (2.6) holds.

PROOF. Immediate.

3. L-SIMPLE SPACES.

DEFINITION 3.1. Let $(V, || \cdot ||)$ be a normed space, and $G \in \Delta^+ \setminus \{\varepsilon_0, \varepsilon_\infty\}$. Then $(V, || \cdot ||, G)$, the L-simple space generated by $(V, || \cdot ||)$ and G , is the pair

(V, ν) in which $\nu: V \rightarrow \Delta^+$ is defined by

$$\nu(p) = \|p\|_{\otimes_L G}, p \in V. \quad (3.1)$$

In Particular, Sum-simple spaces are also simple spaces.

DEFINITION 3.2. Let $\mathcal{G}$ be a class of pairs (V, ν) in which V is a vector space, and $\nu: V \rightarrow \Delta^+$ satisfies (2.1), (2.4), and τ is a triangle function. If for any $(V, \nu) \in \mathcal{G}$ it holds that

$$\nu(p+q) \geq \tau(\nu(p), \nu(q)), p, q \in V, \quad (3.2)$$

then τ is said to be universal for $\mathcal{G}$.

DEFINITION 3.3. If $F, G \in \Delta^+$ and there exists $\alpha \in (0, \infty)$ such that $G = \alpha \otimes_L F$, then F and G are said to be L -comparable. We write

$$\mathcal{G}_L(\Delta^+) = \{(F, G) \in \Delta^+ \times \Delta^+ \mid F, G \text{ are } L\text{-comparable}\}.$$

THEOREM 3.1. (cf. Theorem 8.4.2, 8.4.4 and Problem 8.8.1 in [5]). Triangle function τ is universal for the class $\mathcal{G}_L$ of all L -simple spaces if and only if $\tau|_{C_{L(\Delta^+)}} \leq \tau_{M, L}|_{C_{L(\Delta^+)}}$.

PROOF. ($\Leftarrow$) First, we show that $\tau_{M, L}$ is universal for $\mathcal{G}_L$. In fact, if (V, ν) is a L -simple space, then for all $p \in V, y \in I$,

$$\begin{aligned} \nu(p)^\wedge(y) &= \text{Sup}\{x \mid \nu(p)(x) < y\} \\ &= \text{Sup}\{x \mid G(x \otimes_L \|p\|) < y\} \\ &= \text{Sup}\{\|p\| \otimes_L \zeta \mid G(\zeta) < y\} \\ &= \|p\| \otimes_L \text{Sup}\{\zeta \mid G(\zeta) < y\} \\ &= \|p\| \otimes_L G(y). \end{aligned}$$

Therefore, from Lemma 1.1. (iii), we obtain that for all $p, q \in V, y \in I$,

$$\begin{aligned} \nu(p+q)^\wedge(y) &= \|p+q\| \otimes_L G(y) \\ &< (\|p\| + \|q\|) \otimes_L G(y) \\ &= L(\|p\| \otimes_L G(y), \|q\| \otimes_L G(y)) \\ &= L(\nu(p)^\wedge(y), \nu(q)^\wedge(y)). \end{aligned}$$

and from (7.7.10) in [5], we have

$$\nu(p+q) \geq [L(\nu(p)^\wedge, \nu(q)^\wedge)] = \tau_{M, L}(\nu(p), \nu(q)).$$

In general, if $\tau|_{C_{L(\Delta^+)}} \leq \tau_{M, L}|_{C_{L(\Delta^+)}}$, then for any L -simple space (V, ν) and for any $p, q \in V$,

$$\tau(\nu(p), \nu(q)) \leq \tau_{M, L}(\nu(p), \nu(q)).$$

In fact, if $p = 0$, or $q = 0$, then $\tau(\nu(p), \nu(q)) = \nu(q)$ or $\nu(p)$, and if

$p \neq 0$ and $q \neq 0$, then $\|q\|/\|p\| \in (0, \infty)$ and $\nu(q) = \|q\|/\|p\| \otimes_L \nu(p)$, i.e.,

$(\nu(p), \nu(q)) \in C_{L(\Delta^+)}$. Consequently, τ is universal for $\mathcal{G}_L$ because so is $\tau_{M, L}$.

($\Rightarrow$). If $\tau|_{C_{L(\Delta^+)}} \leq \tau_{M, L}|_{C_{L(\Delta^+)}}$ does not hold, then there exists a pair

$(G, F) \in C_{L(\Delta^+)}$ such that $\tau(G, F) \not\leq \tau_{M, L}(G, F)$. Because $\tau(\varepsilon_0, F) = \tau_{M, L}(\varepsilon_0, F) = F$, we have

and $\tau(\varepsilon_\infty, F) = \varepsilon_\infty$, that $G \notin \{\varepsilon_0, \varepsilon_\infty\}$. Now, we consider the L -simple space

$(R, \nu) = (R, \|\cdot\|, G)$ generated by the real line with the usual norm and G . Since

$(G, F) \in \mathcal{C}_L^+(\Delta^+)$, there exists $\alpha \in (0, \infty)$ such that $F = \mathcal{M}_L G$. In addition, from $\tau(G, F) < \tau_{M, L}(G, F)$, we know that there exists $x_0 \in (0, \infty)$ such that

$$\tau(G, F)(x_0) > \tau_{M, L}(G, F)(x_0), \text{ and furthermore}$$

$$\tau(v(1), v(\alpha))(x_0) = \tau(v(1), \mathcal{M}_L v(1))(x_0)$$

$$= \tau(G, F)(x_0)$$

$$> \tau_{M, L}(G, F)(x_0)$$

$$= \sup \{M(G(u), F(v)) \mid L(u, v) = x_0\}$$

$$> M(G(x_0 \delta_L(1+\alpha)), F(\alpha^*_L(x_0 \delta_L(1+\alpha))))$$

$$= G(x_0 \delta_L(1+\alpha))$$

$$= v(1+\alpha)(x_0)$$

because

$$L(x_0 \delta_L(1+\alpha)), \alpha^*_L(x_0 \delta_L(1+\alpha)) = (1+\alpha) *_{L, L}(x_0 \delta_L(1+\alpha)) = x_0$$

and

$$F(\alpha^*_L(x_0 \delta_L(1+\alpha))) = G((\alpha^*_L(x_0 \delta_L(1+\alpha))) \delta_L \alpha)$$

$$= G(x_0 \delta_L(1+\alpha)).$$

This contradicts (3.2).

COROLLARY 3.1. Any L -simple space is a PN space under $\tau_{T, L}$.

Let $(V, || \cdot ||)$ be a normed space, $\alpha \in (0, \infty)$ and $G \in \Delta^+ \setminus \{\epsilon_0, \epsilon_\infty\}$. By the α -simple space generated by $(V, || \cdot ||)$ and G , $(V, || \cdot ||, G, \alpha)$, we mean the pair (V, v) in which $v: V \rightarrow \Delta^+$ is defined by $v(p) = G(\delta/||p||^\alpha)$, $p \in V$.

The following corollary characterizes α -simple spaces.

COROLLARY 3.2. (cf. Problem 8.8.2 in [5]) For $\alpha \in (0, \infty)$, any α -simple space is a PN space under $\tau_{M, K_{1/\alpha}}$.

PROOF. It is sufficient to note that $x \delta_{K_{1/\alpha}} a = x/a^\alpha$, $x, a \in (0, \infty)$, and so α -simple spaces are also $K_{1/\alpha}$ -simple spaces.

REFERENCES

1. ŠERSTNEV, A.N., Random normed spaces: problem of completeness, Kazan. Gos. University Učen. Zap., 122(1962), 3-20.
2. ŠERSTNEV, A.N., On the notion of a random normed space, Dokl. Akad. Nauk SSSR, 149(1963), 280-283.
3. ŠERSTNEV, A.N., Some best approximation problems in random spaces, Dokl. Akad. Nauk SSSR, 149(1963), 539-542.
4. ŠERSTNEV, A.N., Some best approximation problems in random spaces, Rev. Roumaine Mathematics Pures Application, 9(1964), 771-789.
5. SCHWEIZER, B. and SKLAR, A., Probabilistic Metric Spaces, North Holland (Amsterdam, 1983).
6. MUŠTARI D.H. and ŠERSTNEV, A.N., A problem about triangle inequalities for random normed spaces, Kazan. Gos. University Učen. Zap., 125(1965), 102-113.
7. MUŠTARI, D.H. and ŠERSTNEV, A.N., Les fonctions du triangle pour les espaces normés aléatoires, in General Inequalities 1, (E.F. Beckenbach, ed.), Birkhäuser Verlag (Basel, 1978), 255-260.