A NOTE ON ANALYTIC MEASURES

NAKHLE ASMAR

Department of Mathematics and Computer Science
California State University, Long Beach
Long Beach, California 90840

(Received March 24, 1988 and in revised form April 27, 1988)

ABSTRACT: Let G be a compact Abelian group with character group X . Let S be a subset of X such that, for some real-valued homomorphism ψ on X , the set $S \cap \psi^{-1}(]-\infty, \psi(\chi)])$ is finite for all χ in X . Suppose that μ is a measure in $M(G)$ such that $\hat{\mu}$ vanishes off of S , then μ is absolutely continuous with respect to the Haar measure on G .

KEY WORDS AND PHRASES. *Analytic measures, absolutely continuous, Bochner's Theorem.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 43A05 43A17

1. INTRODUCTION.

Let G denote a compact Abelian group with character group X . Suppose that ψ is a real-valued homomorphism on X , and let ϕ denote the adjoint homomorphism of ψ . Thus ϕ is the continuous homomorphism from $\mathbf{R}$ into G such that the identity $\chi \circ \phi(r) = \exp(i\psi(\chi)r)$ holds for all r in $\mathbf{R}$, and all χ in X . We denote by $M(G)$ the linear space of all complex-valued regular Borel measures on G . In the terminology of de Leew and Glicksberg [1], a measure μ in $M(G)$ is called *ϕ -analytic* if its Fourier transform $\hat{\mu}$ vanishes on $\{\chi \in X: \psi(\chi) < 0\}$.

Suppose that S is a nonvoid subset of X . Let $M_S(G)$ denote the closed linear subspace of $M(G)$ consisting of the measures μ with $\hat{\mu}$ vanishing off of S . The set S will be called a *B-set* (B for Bochner) if there is a nonzero homomorphism ψ from X into $\mathbf{R}$ such that the set $S \cap \psi^{-1}(]-\infty, \psi(\chi)])$ is finite for all χ in X . The homomorphism ψ may depend on S , and may not be unique. For example, a sector with opening less than π in the lattice plane $\mathbf{Z} \times \mathbf{Z}$ is a B-set. The first orthant in $\mathbf{Z}^\omega$ (the weak direct product of countably many copies of $\mathbf{Z}$) is also a B-set. Once we have chosen a homomorphism ψ , we will refer to S as a B-set with respect to the homomorphism ψ .

A theorem due to Bochner [2], on $\mathbf{T}^2$, the two-dimensional torus, asserts that if $\mu \in M(\mathbf{T}^2)$ is such that $\hat{\mu}$ vanishes off of a sector of opening less than π , then μ is absolutely continuous. (The expression "absolutely continuous" will always mean absolutely continuous with respect to the Haar measure on the group in consideration.) A generalization of this result is given in de Leew and Glicksberg [1], Theorem (3.4).

It is easy to construct B-sets in $\mathbf{Z} \times \mathbf{Z}$ that are contained in no sector with opening less than π . For example, consider the set $S = \{(x, y) \in \mathbf{Z} \times \mathbf{Z}: y \geq \log(1 + |x|)\}$. Using results from [1], we will show that the conclusion of Bochner's theorem holds for B-sets. We have the following theorem.

(1.1) **THEOREM.** Let S be a B-set in X . Suppose that μ is in $M_S(G)$, then μ is absolutely continuous.

Before proving the theorem we make a few observations. Suppose that S is a B -set, with respect to some homomorphism ψ . Clearly, there is a character χ_0 in S such that $\psi(\chi_0) \leq \psi(\chi)$ for all χ in S . Note that any translate of S by an element of X is also a B -set with respect to the same homomorphism ψ . Hence by shifting S by $-\chi_0$, if necessary, we may suppose that $\psi(\chi) \geq 0$ for all χ in S . In this case, given a measure μ in $M_S(G)$, we consider the measure $\bar{\chi}_0\mu$ which is in $M_{S-\chi_0}(G)$. The set $S-\chi_0$ is a B -set, with respect to the homomorphism ψ ; and $\bar{\chi}_0\mu$ is absolutely continuous if and only if μ is.

If μ is in $M(G)$, we write μ_a and μ_s to denote its absolutely continuous part and its singular part respectively.

(1.2) Lemma. Let S be a B -set in X . Suppose that μ is in $M_S(G)$, then μ_a and μ_s are in $M_S(G)$.

Proof. As we observed before the lemma, we may suppose that $\psi(S) \subseteq [0, \infty[$. Let ϕ denote the adjoint homomorphism of ψ , and let χ_1 be an arbitrary character in $X \setminus S$, the complement of S in X . We want to show that

$$(1) \quad \dot{\mu}_s(\chi_1) = \dot{\mu}_a(\chi_1) = 0.$$

First, note that if S is finite then $\mu = \mu_a$, and the lemma is obviously true. So suppose for the rest of the proof that S is infinite. Let χ_2 in X be such that $\psi(\chi_1) < \psi(\chi_2)$. Let $A = \{\chi \in X: \psi(\chi) < \psi(\chi_2)\} \cap \text{supp } \mu$. The set A is either void or finite. Define the measure σ in $M(G)$ by,

$$\sigma = \mu - \sum_{\chi \in A} \dot{\mu}(\chi)\chi,$$

where the above sum is 0 if A is empty. We have

$$\hat{\sigma}(\chi) = \begin{cases} \dot{\mu}(\chi) & \text{if } \chi \notin A; \\ 0 & \text{if } \chi \in A. \end{cases}$$

Hence $\hat{\sigma}$ vanishes off of $\psi^{-1}([\psi(\chi_2), \infty]) \cap S$, which implies that σ is ϕ -analytic. It follows from [1], the Main Theorem, Proposition (2.3.2), and Theorem (5.1), that $\hat{\sigma}_a$ and $\hat{\sigma}_s$ vanish off of $\psi^{-1}([\psi(\chi_2), \infty]) \cap S$. Since $\mu_s = \sigma_s$, it follows that $\dot{\mu}_s$ vanishes off of $\psi^{-1}([\psi(\chi_2), \infty]) \cap S$. Therefore, $\dot{\mu}_s(\chi_1) = 0$, and the lemma follows. $\square$

Proof of Theorem (1.1). According to Lemma (1.2), it is enough to show that $\dot{\mu}_s(\chi) = 0$ for all χ in S . The proof is by contradiction. Assume that $\dot{\mu}_s(\chi_0) \neq 0$ for some χ_0 in S . Let χ_1 in X be such that $\psi(\chi_1) > \psi(\chi_0)$. (Here also we are assuming that S is infinite and $\psi(S) \subseteq [0, \infty[$.) Let $A = \{\chi \in X: \psi(\chi) \leq \psi(\chi_1), \text{ and } \dot{\mu}_s(\chi) \neq 0\}$. Then A is contained in $\psi^{-1}([-\infty, \psi(\chi_1)]) \cap S$; and so A is finite and χ_0 is in A . Define the measure ν in $M(G)$ by

$$\nu = \mu_s - \sum_{\chi \in A} \dot{\mu}_s(\chi)\chi.$$

We have

$$\dot{\nu}(\chi) = \begin{cases} \dot{\mu}_s(\chi) & \text{if } \chi \notin A; \\ 0 & \text{if } \chi \in A. \end{cases}$$

Thus $\dot{\nu}$ vanishes off of $\psi^{-1}([\psi(\chi_1), \infty]) \cap S$, and hence it is ϕ -analytic. Applying Proposition (5.1), [1], we see that $\dot{\nu}_s$ and $\dot{\nu}_a$ vanish off of $\psi^{-1}([\psi(\chi_1), \infty]) \cap S$. Since $\nu_s = \mu_s$, it follows that $\dot{\mu}_s$ vanishes off of $\psi^{-1}([\psi(\chi_1), \infty]) \cap S$. This is plainly a contradiction since $\psi(\chi_0) < \psi(\chi_1)$, and by assumption $\dot{\mu}_s(\chi_0) \neq 0$. $\square$

REFERENCES

1. de Leew, K., and I. Glicksberg. Quasi-Invariance and analyticity of measures on compact groups. Acta. Math. **103** 1963, 179-205
2. Bochner, S. Boundary values of analytic functions in several variables and almost periodic functions. Ann. of Math. **45** 1944, 708-722