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ON WEAKER FORMS OF COMPACTNESS LINDELÖFNESS AND COUNTABLE COMPACTNESS

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ABSTRACT. A theory of e-countable compactness and e-Lindelöfness which are weaker than the concepts of countable compactness and Lindelöfness respectively is developed. Amongst other results we show that an e-countably compact space is pseudocompact, and an example of a space which is pseudocompact but not e-countably compact with respect to any dense set is presented. We also show that every e-Lindelöf metric space is separable.

KEY WORDS AND PHRASES. e-compact, e-countably compact, e-Lindelöf, pseudocompact, separable.
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1. INTRODUCTION.

Using the terminology and notation in [3], it is known that a topological space X is compact iff its enlargement *X contains only near-standard points, and that a subset A of a regular Hausdorff space is relatively compact iff *A contains only near standard points. Hechler [1] wanted to know what this condition implied topologically in not necessarily regular spaces. He was led to the notion of what he called 'e-compactness' which is weaker than the notion of compactness. It is the purpose of this paper to develop a theory on the analogous concepts of e-countable compactness and e-Lindelöfness in the spirit of Hechler's study of e-compactness [1]. In particular we extend the well-known result that a countably compact space is pseudocompact to an e-countably compact space is pseudocompact. We also show that the Lindelöf condition in the theorem that every Lindelöf metric space is separable can be weakened to e-Lindelöf.

2. PRELIMINARIES.

We begin by recalling Hechler's definition of e-compactness and e-regularity.

Let D be a dense subspace of a topological space X.

DEFINITION 2.1. ([1]): (a) X is said to be e-compact with respect to D if each open cover of X contains a finite subcollection that covers D. (b) X is e-regular with respect to a dense subset D if for each closed $F \subset X$ and each $p \notin F$ there exist disjoint open sets U and V such that $p \in U$ and $F \cap D \subset V$. By analogy we introduce the following:

DEFINITION 2.2. X is e-countably compact with respect to D if every countable open cover of X has a finite subcollection covering D.

DEFINITION 2.3. X is e-Lindelöf with respect to D if every open cover of X has a countable subcollection covering D.

Recall Hechler's extension of the topology T for X:
Let $\mathcal{E}$ be a family of subsets of X, and let $T(\mathcal{E}) = \{U-F \mid U \in T$ and F is a subset of a finite union of members of $\mathcal{E}\}$. By $X(\mathcal{E})$ we shall mean X with the extended topology $T(\mathcal{E})$. Hechler used the construction of the extended topology $T(\mathcal{E})$ to provide examples of spaces which are e-compact but not compact (see [1] p. 223).

3. RESULTS.

The following two theorems are analogous to the corresponding theorems on e-compactness (see [1]).

THEOREM 3.1. Let X be Lindelöf. Then $X(\mathcal{E})$ is e-Lindelöf iff there exists a dense set D (with respect to $X(\mathcal{E})$) such that for every $E \in \mathcal{E}$, $E \cap D$ is countable.

PROOF. Suppose $X(\mathcal{E})$ is e-Lindelöf with respect to a dense set D. Assume for some $E \in \mathcal{E}$, $E \cap D$ is uncountable. Then $\{X - (E-\{x\}) \mid x \in E\}$ is an open cover of X having no countable subcover of D, contrary to $X(\mathcal{E})$ being e-Lindelöf. Thus $E \cap D$ is countable for all $E \in \mathcal{E}$.

Conversely suppose $E \cap D$ is countable for all $E \in \mathcal{E}$ and D is dense with respect to $X(\mathcal{E})$. Suppose $\{U_\alpha - F_\alpha : \alpha \in \Lambda\}$ is an open cover of X. Then $\{U_\alpha : \alpha \in \Lambda\}$ covers X, and as X is Lindelöf, there is a countable subcollection $\{U_{\alpha_i}\}_{i=1}^{\infty}$, say, covering X. Now $\{U_{\alpha_i} - F_{\alpha_i}\}_{i=1}^{\infty}$ will cover all except at most countably many points $\{x_j\}_{j=1}^{\infty}$ of D. But $\{x_j\}_{j=1}^{\infty} \subset \cup \mathcal{B}$, where $\mathcal{B}$ is a countable subcollection of $\{U_\alpha - F_\alpha : \alpha \in \Lambda\}$. Thus $\{U_{\alpha_i} - F_{\alpha_i}\}_{i=1}^{\infty} \cup \mathcal{B}$ is a countable subcollection of $\{U_\alpha - F_\alpha : \alpha \in \Lambda\}$ covering D, showing that $X(\mathcal{E})$ is e-Lindelöf.

THEOREM 3.2. Let X be countably compact. Then $X(\mathcal{E})$ is e-countably compact iff there exists a dense subset D (with respect to $X(\mathcal{E})$) such that for every set $E \in \mathcal{E}$, $E \cap D$ is finite.

PROOF. Suppose $X(\mathcal{E})$ is e-countably compact with respect to D. Assume for some $E \in \mathcal{E}$, $E \cap D$ is infinite. Choose an infinite sequence $\{x_n : n \in \mathbb{N}\}$ in $E \cap D$ and let $F = \{x_n : n \in \mathbb{N}\}$. Then $\{X - (F-\{x\}) : x \in F\}$ is a countable open cover of X with no finite subcover of D, contrary to X being e-countably compact.

Conversely suppose $E \cap D$ is finite for all $E \in \mathcal{E}$, and D a dense set with respect to $X(\mathcal{E})$. Suppose $\{U_i - F_i\}_{i=1}^{\infty}$ is a countable open cover of $X(\mathcal{E})$. Then $\{U_i\}_{i=1}^{\infty}$ covers X, and as X is countably compact there is a finite subcover $\{U_{k_i}\}_{i=1}^{r}$ of X. Now $\{U_{k_i} - F_{k_i}\}_{i=1}^{r}$ covers all except at most finitely many points of D, $\{x_j\}_{j=1}^{t}$, say. But $\{x_j\}_{j=1}^{t} \subset \cup \mathcal{S}$ where $\mathcal{S}$ is a finite subcollection of $\{U_i - F_i\}_{i=1}^{\infty}$. Thus $\{U_{k_i} - F_{k_i}\}_{i=1}^{r} \cup \mathcal{S}$ is a finite subcollection of $\{U_i - F_i\}_{i=1}^{\infty}$ covering D, showing $X(\mathcal{E})$ is e-countably compact.

THEOREM 3.3. If X is e-countably compact, then X is pseudocompact.

PROOF. Suppose X is e-countably compact with respect to D. Let f be a continuous real valued function of X. Then if $G_n = \{x \in X \mid |f(x)| < n\}$, $\{G_n : n \in \mathbb{N}\}$ would be an open cover of X having a finite subcover of D. Clearly then $D \subset G_n$ for some $n \in \mathbb{N}$. By continuity of f, we have $f(X) \subset [-n,n]$, showing f is bounded. It is well known that a countably compact first countable Hausdorff space is regular. We have the following:

THEOREM 3.4. Every e-countably compact first countable Hausdorff space is e-regular (with respect to a dense set D).

PROOF. Let $p \notin F$, F closed in X. Let $\mathcal{B}$ be a countable open neighbourhood base at p. Since X is Hausdorff, for each $q \in F$ there exists open G_q and $B_q \in \mathcal{B}$ such that $G_q \cap B_q = \phi$. Let $\mathcal{B}' = \{B_q\}_{q \in F}$, which, being a subfamily of $\mathcal{B}$, must be countable. For each $B \in \mathcal{B}'$ let $H_B = \cup\{G_q \mid G_q \cap B = \phi\}$. Then $\{H_B \mid B \in \mathcal{B}'\}$ is a countable family of open sets covering F so that $\{X-F\} \cup \{H_B \mid B \in \mathcal{B}'\}$ covers X. Since X is e-countably compact there exists a finite family $\mathcal{B}'' \subset \mathcal{B}'$ such that $D \subset (X-F) \cup \cup\{H_B : B \in \mathcal{B}''\}$. Thus $F \cap D \subset \cup\{H_B : B \in \mathcal{B}''\} = V$. Let $U = \cap\{B : B \in \mathcal{B}''\}$ which is open and contains p. It is easily verified that $U \cap V = \phi$.

We now show how separability relates to the generalizations of compactness and Lindelöfness introduced above. It is well known that every Lindelöf metric space is separable. In fact the Lindelöf condition can be weakened to e-Lindelöfness as the following result shows.

THEOREM 3.5. If X is metric and e-Lindelöf with respect to a dense set D then X is separable.

PROOF. For each $n \in \mathbb{N}$ let $U_n = \{S(x, 1/n) \mid x \in X\}$. Since U_n is an open cover of X, there exists a countable set $F_n \subset X$ such that $D \subset \cup\{S(x, 1/n) : x \in F_n\}$. Let $F = \cup\{F_n : n \in \mathbb{N}\}$. Then F is countable.

CLAIM. F is dense in X: Let $y \in X$ be arbitrary, $\varepsilon > 0$. Find N such that $1/N < \varepsilon$. Now $S(y, 1/2N) \cap D \neq \phi$ so there exists $z \in D$ such that $d(y,z) < 1/2N$. Now $D \subset \cup \{S(x,1/2N): x \in F_{2N}\}$ implies that there exists $x \in F_{2N} \subset F$ such that $d(z,x) < 1/2N$. Thus $d(y,x) < d(y,z) + d(z,x) < 1/2N + 1/2N = 1/N < \varepsilon$.

Hence $S(y,\varepsilon) \cap F \neq \phi$, showing F dense in X.

We then have:

THEOREM 3.6. For metric spaces the following are equivalent

(a) X is separable

(b) X is 2nd countable

(c) X is Lindelöf

(d) X is e-Lindelöf (with respect to any dense set)

Since every e-countably compact space is pseudocompact and every pseudocompact T_4 space is countably compact we also have

THEOREM 3.7. For T_4 spaces, the following are equivalent

(a) X is countably compact

(b) X is e-countably compact (with respect to any dense set)

(c) X is pseudocompact

4. EXAMPLES.

This is an example of a space which is pseudocompact but not e-countably compact with respect to any dense set.

Consider Z^+, the positive integers with the relatively prime topology, i.e. with basis $B = \{U_a(b) \mid a,b \in Z^+, (a,b)=1\}$ where $U_a(b) = \{b + na \in Z^+ \mid n \in Z\}$. We shall show that there exists a countable open cover $\mathcal{U}$ of Z^+ such that the closures of no finite subcollection of $\mathcal{U}$ covers Z^+. This would then imply that Z^+ is not e-countably compact with respect to any dense set. Recall that Z^+ is pseudocompact ([2], p.83). Let then $\mathcal{U} = \{U_3(1), U_5(2), U_5(3), U_5(4), U_7(5), U_7(6), U_{11}(7), U_{11}(8), \ldots\}$, and let $\mathcal{U}' = \{U_{p_i}(\alpha_i)\}_{i=1}^n$ be any finite subcollection of $\mathcal{U}$.

Case 1: $U_3(1) \notin \mathcal{U}'$. Then let $m = p_1 p_2 \ldots p_n$ and note that

$$U_m(m+1) \cap U_{p_i}(\alpha_i) = \emptyset \ \forall i$$

so that $\cup \overline{\mathcal{U}'} \neq Z^+$.

Case 2: $U_3(1) \in \mathcal{U}'$. If $U_3(1)$ is the only element in $\mathcal{U}'$, then $U_3(2) \cap U_3(1) = \phi$, so that $\cup \overline{\mathcal{U}'} \neq Z^+$; otherwise let $m = p_2 p_3 \ldots p_n$ where $p_1 = 3$, and $p_i \neq 3 \ \forall i$, $2 \leq i \leq m$.

Then

$$U_m(m+1) \cap U_{p_i}(\alpha_i) = \phi \ \forall i, \ 2 \leq i \leq n$$

and $U_3(m+1) \cap U_3(1) = \phi$ (in the case where $3 \nmid m+1$) so that $U_m(m+1) \cap U_3(m+1)$ is a neighbourhood of m+1 meeting no member of $\mathcal{U}'$. If $3 \nmid m+1$, let

$$m' = p_o m$$

where p_o is a prime $\ni 3 \nmid p_o - 1$ and $p_o \neq 3$. Then $3 \nmid p_o m$ and $3 \nmid p_o m + 1$.
Hence $U_{m'}(m'+1) \cap U_3(m'+1)$ is the required neighbourhood of $m'+1$ meeting no member of $\mathcal{P}'$.

This example is motivated by the result that a Lindelöf countably compact space is compact. The analogous statement that an e-Lindelöf, e-countably compact space is e-compact is not in general true as the following example shows:
Recall the Novak space (see [2] p. 134): Let Z^+ denote the positive integers with the discrete topology and S the Stone-Cech compactification βZ^+ of Z^+. Let F be the family of all countably infinite subsets of S, well-ordered by the least ordinal Γ of cardinal 2^c = card (S). Let $\{P_A | A \in F\}$ be a collection of subsets of S such that card $(P_A) < 2^c$, $P_D \subset P_A$ whenever $D < A$, and $\hat{f}(P_A) \cap P_A = \phi$ where $\hat{f}$ is the unique extension to S of the continuous function $f: Z^+ \to Z^+$ which permutes each odd integer with its even successor, i.e. $f(n) = n + (-1)^{n+1}$. Then we define

$P = \cup \{P_A | A \in F\}$, and then define Novak's space by

$$X = P \cup Z^+.$$

Note that $c\ell_X(Z^+) = X$ ([2] p. 135), hence X is e-Lindelöf with respect to Z^+. Also X is countably compact ([2] p. 135), but as X is not compact, X cannot be absolutely closed (as a regular absolutely closed space is compact). Thus X is not e-compact with respect to any dense set.

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