ON THE STRUCTURE OF SUPPORT POINT SETS

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ABSTRACT. Let $X$ be a metrizable compact convex subset of a locally convex space. Using Choquet's Theorem, we determine the structure of the support point set of $X$ when $X$ has countably many extreme points. We also characterize the support points of certain families of analytic functions.

KEY WORDS AND PHRASES: Support point, Extreme point, Choquet's Theorem.


1. INTRODUCTION.

Let $X$ be a subset of a locally convex space $E$. A continuous linear functional $J$ on $X$ is said to be associated with $f \in X$ if $\text{Re } J(f) = \max \{ \text{Re } J(g) : g \in X \}$ and $\text{Re } J$ is non constant on $X$. In this case we call $f$ a support point of $X$. The set of support points of $X$ will be denoted by $\text{Supp } X$. The set of extreme points of a convex subset $F$ of $E$ will be denoted by $\text{Ext } F$.

Let $D = \{ z : |z| < 1, z \in \mathbb{C} \}$ and equip the space $A$ of functions analytic in $D$ with the topology of uniform convergence on compact subsets of $D$. This topology is metrizable [1, p.1]. Every continuous linear functional $J$ on $A$ is induced by a sequence $(b_n)_{n=0}^{\infty}$ which satisfies $\limsup_{n} |b_n|^{1/n} < 1$ and $J(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A$ [1, p.36]. Recently, the support points of many subclasses of $A$ have been studied. For more details see [1] and [2].

In Section 2, we consider a metrizable compact convex set $X$ in a locally convex space. Using Choquet's theorem we determine the structure of $\text{Supp } X$ when $\text{Ext } X$ is countable (Theorem 2.1).
In Section 3, we consider the classes: \( P(p) = \{ f(z) = \sum_{n=1}^{\infty} a_n z^n \in A : \sum_{n=1}^{\infty} |a_n|^p \leq 1 \} , 1 \leq p < \infty \). In Theorem 3.4, we determine \( \text{Supp } P(p) \). Indeed, it is shown that \( \text{Supp } X \) is in 1-1 correspondence with a proper subset of \( \text{Supp Ball}(\ell_p) \).

2. SUPPORT POINTS OF SETS WITH COUNTABLY MANY EXTREME POINTS.

Let \( E \) be a locally convex space, and suppose that \( X \) is a metrizable compact convex subset of \( E \). A theorem by Choquet [3, p.19] says that if \( x \in X \) then there exists a probability measure \( \mu_x \) on \( X \), supported by \( \text{Ext } X \), such that \( L(x) = \int L \, d\mu_x \) for every \( L \in E^\ast \). In case \( \text{Ext } X \) is countable (possibly finite), we have the following:

CHOQUET'S THEOREM (Countable Case). Suppose \( \text{Ext } X = \{ f_n \} \) is countable. Then \( X = \{ \sum_{n} \lambda_n f_n : \lambda_n \geq 0 \) for each \( n \) and \( \sum_{n} \lambda_n = 1 \).

**Proof.** Let \( f \in X \). By Choquet's Theorem, there exists a probability measure \( \mu_f \) on \( X \), supported by \( \{ f_n \} \), such that \( L(f) = \int L \, d\mu_f \). Thus \( L(f) = \sum_{n} \mu(f_n) L(f_n) \). Hence \( L(f - \sum_{n} \mu(f_n)f_n) = 0 \).

Since this is true for every \( L \in E^\ast \), we get \( f = \sum_{n} \mu(f_n) f_n \), as required.

We proceed to the main result of this section.

**Theorem 2.1.** Let \( X \) be a metrizable compact convex subset of a locally compact space \( E \) such that \( \text{Ext } X = \{ f_n \} \) is countable. For each positive integer \( n \), set \( K_n \) equal to the closed convex hull of \( \{ f_i : i \neq n \} \). Then

1. \( \text{Supp } X \) is contained in the union of those \( K_n \) which are proper subsets of \( X \).

2. \( K_n \subseteq \text{Supp } X \) if and only if \( f_n \notin \text{closed affine hull of } \{ f_i : i \neq n \} \).

**Proof.** To prove (1), let \( f \in \text{Supp } X \). By Choquet's Theorem, we can write \( f = \sum_{i} \lambda_i f_i \) with each \( \lambda_i \geq 0 \) and \( \sum_{i} \lambda_i = 1 \). Let \( \phi \) be a continuous linear functional associated with \( f \). Then \( \text{Re } \phi(f) = \sum_{i} \lambda_i \text{Re } \phi(f_i) \leq \sum_{i} \lambda_i \text{Re } \phi(f) = \text{Re } \phi(f) \). Hence we must have \( \text{Re } \phi(f_i) = \text{Re } \phi(f) \) whenever \( \lambda_i > 0 \). On the other hand, since \( \text{Re } \phi \) is non-constant on \( X \), we must have \( \text{Re } \phi(f_i) \neq \text{Re } \phi(f) \) for some \( i \). We conclude that \( \lambda_i = 0 \) for some \( i \), as required.

To prove (2), suppose that \( f_n \) does not belong to the closed affine hull \( H \) of \( \{ f_i : i \neq n \} \) and fix \( g \in K_n \). Then \( H \cdot g \) is a closed real subspace of \( E \) not containing \( f_n - g \). A version of the Hahn-Banach theorem [4, page 59] gives a functional \( J \) in \( E^\ast \) whose real part \( \phi \) vanishes on \( H \cdot g \) while \( \phi(f_n - g) = -1 \). Set \( \phi(f_{n+1}) = b \). Then \( \phi(f_n) = b - 1 \) while \( \phi(f_i) = \phi(f_{n+1}) = b \) for every \( i \neq n \). Thus, \( \phi(g) = b \) for all \( g \in K_n \). For any \( h \in X \), by Choquet's Theorem, we have \( h = \sum_{i} \beta_i f_i \) with \( \beta_i \geq 0 \) and \( \sum_{i} \beta_i = 1 \). Thus \( \phi(h) = \beta_n(b - 1) + \sum_{i \neq n} \beta_i b = b \cdot \beta_n \leq b \). This shows that \( g \in \text{Supp } X \).

Conversely, assume that \( K_n \subseteq \text{Supp } X \). For ease of notation we take \( n = 1 \) and assume \( \text{Ext } X = \{ f_n \}_{n=1}^{\infty} \) is infinite. By assumption, \( f = \sum_{i=2}^{\infty} 1 \cdot f_i \) is a support point of \( X \). Let \( \phi \) be an associated linear functional in \( E^\ast \) and set \( S = \{ g \in E : \text{Re } \phi(g) = \text{Re } \phi(f) \} \). Note that \( S \) is a closed affine subspace of \( E \). Since \( \text{Re } \phi(f) = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \text{Re } \phi(f_i) \leq \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \text{Re } \phi(f) = \text{Re } \phi(f) \), we have \( \text{Re } \phi(f_i) = \text{Re } \phi(f) \) for all \( i \geq 2 \). Thus the closed affine hull
of \( \{f_i : i \neq 1\} \subseteq S \). On the other hand, in view of Choquet's Theorem, if \( f_1 \in S \) then \( \Re \phi \) would be constant on \( X \).

Thus \( f_1 \notin S \) and consequently, \( f_1 \notin \text{closed affine hull of } \{f_i : i \neq 1\} \).

**EXAMPLES.** (1) Let \( X \) be a triangle in \( \mathbb{R}^2 \) with vertices \( f_1, f_2, f_3 \). These vertices are the extreme points of \( X \) and the affine hull of any two of them is a line, not containing the third. The theorem guarantees that \( \text{Supp} \ X = \bigcup_{n=1}^{3} K_n \), which is indeed the boundary of \( X \).

(2) Let \( X \) be a square in \( \mathbb{R}^2 \) with vertices \( f_1, f_2, f_3, f_4 \). The affine hull of any three of the \( f_i \)'s is all of \( \mathbb{R}^2 \). In particular, each \( f_i \in \text{affine hull of } \{f_j : j \neq i\} \). The theorem guarantees that no \( K_n \) is contained in \( \text{Supp} \ X \).

In fact, \( \text{Supp} \ X = \text{the boundary of } X \) has no interior.

(3) Let \( T \) be the family of all functions which are analytic and univalent in \( D \), and take the form \( f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), \( a_n \geq 0 \). By [5], \( \text{Ext} \ T = \{f_n\}_{n=1}^{\infty} \), where \( f_1(z) = z \) and \( f_n(z) = z - \frac{1}{n} z^n \) for \( n > 1 \). For \( n > 1 \), it is clear that \( f_n \) does not belong to the closed affine hull of the remaining \( \{f_j\} \), so \( \bigcup_{n=2}^{\infty} K_n \subseteq \text{Supp} \ X \) by the second part of the Theorem. Since \( f_1 \) is a limit point of the remaining \( f_j \)'s, \( K_1 = X \) and \( \text{Supp} \ X = \bigcup_{n=2}^{\infty} K_n \) by the first part of the Theorem.

**COROLLARY 2.2.** Let \( X \) be as in Theorem 2.1. Then \( \text{Supp} \ X = \bigcup_{\alpha} E_\alpha \), where each \( E_\alpha \) is a subset of \( \text{Ext} \ X \).

**PROOF.** Suppose \( f \in \text{Supp} \ X \) and \( \phi \) is an associated linear functional with \( f \). Writing \( f = \sum \lambda_i f_i \), we see that \( \Re \phi(f) = \Re \phi(f_i) \) whenever \( \lambda_i \neq 0 \). Take \( E_\alpha = \{f_i : \lambda_i \neq 0\} \). Then \( f \in E_\alpha \subseteq \text{Supp} \ X \).

The theorem says these \( E_\alpha \) are proper subsets of \( \text{Ext} \ X \), i.e., they cannot be “too big”. The next proposition implies that they can’t all be singletons, i.e., “too small”.

**PROPOSITION 2.3.** Let \( X \) be a compact convex subset of a locally convex space. If \( X \) has more than two extreme points, then \( \text{Supp} \ X \) is uncountable.

**PROOF.** Without loss of generality we may assume that \( 0 \in X \). Let \( f_1 \) and \( f_2 \) be two independent elements of \( X \), and let \( \phi_1 \) and \( \phi_2 \) be continuous and linear functionals such that \( \phi_1(f_1) = \phi_2(f_2) = 1 \) and \( \phi_1(f_2) = \phi_2(f_1) = 0 \). Define \( \psi : X \to \mathbb{R}^2 \) by \( \psi(f) = (\phi_1(f), \phi_2(f)) \). Then \( \psi(X) \) is a compact convex subset of \( \mathbb{R}^2 \) with non empty interior. Since \( \psi(X) \) has uncountably many boundary points, \( \text{Supp} \psi(X) \) is uncountable. Since \( \psi^{-1} \) takes support points to support points, we see that \( \text{Supp} \ X \) is uncountable too.

**EXAMPLE.** Take \( f_n = e^{-\pi n} \) for \( n = 1, 2, \ldots \) and \( X = \bigcup_{n=1}^{\infty} \{f_n\} \) in \( \mathbb{R}^2 \). Then \( \text{Supp} \ X = \bigcup_{n=1}^{\infty} \bigcup_{n=1}^{\infty} \{f_n, f_{n+1}\} \).

Here all the \( E_\alpha \)'s have cardinality two even though \( \text{Ext} \ X \) is infinite.

**COROLLARY 2.4.** Let \( X \) be as in Theorem 2.1. Then \( \text{Ext} \ X = \text{Supp} \ X \) if and only if \( X \) has two extreme points.
3. SUPPORT POINTS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS.

For \(1 \leq p < \infty\), define \(P(p) = \{ \sum_{n=1}^{\infty} a_n z^n \in A : \sum_{n=1}^{\infty} |a_n|^p \leq 1 \}\). It is easy to see that the classes \(P(p)\) are compact convex subsets of \(A\). These classes are closely related to \(Ball(\ell_p)\) and we will find that \(\text{Supp } P(p)\) is in one-to-one correspondence with a proper subset of \(\text{Supp } Ball(\ell_p)\). As a corollary, we determine the support points of certain families of univalent functions. We use the notation \(a\) for the sequence \(\{a_n\}_{n=1}^{\infty}\).

We begin with a simple observation.

PROPOSITION 3.1. Let \(X\) be the unit ball of a Banach space \(E\). Then \(\text{Supp } X = \{ x \in X : ||x|| = 1 \}\). If \(\phi\) is associated with \(x\), then \(\phi(x) = ||\phi||\).

PROOF. That every vector of norm one belongs to \(\text{Supp } X\) is a consequence of the Hahn-Banach theorem. Suppose conversely that the real part of \(\phi \in X^\ast\) achieves its maximum over \(X\) at \(x\). Since \(X\) is closed under multiplication by scalars of absolute value at most one, we have \(\text{Re } \phi(x) = \sup_{y \in X} \text{Re } \phi(y) = ||\phi||\). Thus \(||\phi|| = \text{Re } \phi(x) \leq ||\phi|| ||x||\) and so \(||x|| = 1\). Moreover \(\text{Re } \phi(x) = ||\phi||\) implies \(\text{Re } \phi(x) \geq |\phi(x)|\), so \(\phi(x)\) is in fact real.

EXAMPLE. The family \(P(p)\) "looks like" the unit ball of \(\ell_p\), but we cannot immediately apply Proposition 3.1 to find its support points. For example, the sequence \(\{a_n\}_{n=1}^{\infty} = \{\sqrt{n}\}_{n=1}^{\infty}\) belongs to the unit sphere of \(\ell_2\), but \(\sum_{n=1}^{\infty} a_n z^n\) is not a support point of \(P(2)\). The problem is that any non-constant linear functional \(\{b_n\}_{n=1}^{\infty} \in \ell_2^\ast\) which assumes its maximum at \(\{a_n\}_{n=1}^{\infty}\) must be a scalar multiple of \(\{a_n\}_{n=1}^{\infty}\). So \(\lim sup n \sqrt{|b_n|} = 1\), which does not correspond to a continuous linear functional on \(A\).

We find the support points of \(P(p)\) by making the remarks in the preceding example more precise.

PROPOSITION 3.2. Suppose \(T : E \to F\) is a linear, injective, and continuous map between topological vector spaces \(E\) and \(F\), and let \(X\) be a subset of \(E\). Then \(Tx \in \text{Supp } TX\) if and only if \(x \in \text{Supp } X\) and some linear functional associated with \(x\) belongs to range \(T^\ast\).

PROOF. Recall that \(T^\ast : F^\ast \to E^\ast\) is defined by \(T^\ast \psi = \psi \circ T\). Suppose \(Tx \in \text{Supp } TX\) and choose \(\psi \in F^\ast\) with \(\text{Re } \psi(Tx) = \max_{y \in X} \text{Re } \psi(Ty)\). Set \(\phi = \psi \circ T\); then \(\psi \in \text{range } T^\ast\), \(\text{Re } \phi(x) = \max_{y \in X} \text{Re } \phi(y)\), and injectivity of \(T\) implies that \(\text{Re } \phi\) is not constant on \(X\).

Conversely, let \(\phi \in \text{range } T^\ast\) such that \(\text{Re } \phi(x) = \max_{y \in X} \text{Re } \phi(y)\). Write \(\phi = \psi \circ T\), \(\psi \in F^\ast\). Then \(\text{Re } \psi(Tx) = \max_{y \in TX} \text{Re } \psi(y)\), and \(\text{Re } \psi\) cannot be constant on \(TX\) since \(\text{Re } \phi\) is not constant on \(X\).

PROPOSITION 3.3. Let \(a \in X = \text{Ball}(\ell_p)\), \((1 < p < \infty)\), with \(||a||_p = 1\), and \(b \in \ell_1\). Then:

1. If \(b\) is associated with \(a\), then there exists \(\beta \neq 0\) with \(\beta |b_n|^q = |a_n|^p\) for all \(n\).
2. If \(b_n = \begin{cases} \frac{a_n}{|a_n|} |a_n|^{p-1} & \text{if } a_n \neq 0 \\ 0 & \text{otherwise} \end{cases}\)

then \(b\) is associated with \(a\).
PROOF. (1) From Proposition 3.1, we learn that \( b(a) = \|b\|_q = \|b\|_1 |a|_p \). Thus we have Hölder equality, so there exists \( \beta \neq 0 \) with \( \beta |a_n|^p = |a_n|^q \) for all \( n \).

(2) \( b(a) = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} |a_n|^{p-1} = \sum_{n=1}^{\infty} |a_n|^p = 1 \), while \( \|b\|_q = \sum_{n=1}^{\infty} |a_n|^{(p-1)q} = 1 \), so this result follows from Hölder’s inequality.

The following is the main result of this section.

THEOREM 3.4. Let \( f(z) = \sum_{n=1}^{\infty} a_n z^n \) be in \( P(p) \). Then \( f \) is a support point of \( P(p) \) if and only if

1. \( f \) is analytic in \( \overline{D} \) and \( \sum_{n=1}^{\infty} |a_n|^p = 1 \), for \( 1 < p < \infty \).
2. \( f(z) = \sum_{n=1}^{N} a_n z^n \), where \( N \) is some positive integer and \( \sum_{n=1}^{N} |a_n| = 1 \) for \( p = 1 \).

PROOF. Define \( T : \ell_p \to \ell_p \) by \( T(a) = \sum_{n=1}^{\infty} a_n z^n \). Clearly \( T \) maps \( \text{Ball}(\ell_p) \) onto \( P(p) \) and \( T \) is injective. Moreover for any \( r < 1 \) and \( a \in \ell_p \), \( (1 < p < \infty) \), we have \( \sup_n |T(a)(r)| = \sum_{n=1}^{\infty} |a_n|^p \leq \|a\|_p (\frac{1}{1-r})^{1/p} \), by Hölder’s inequality, so \( T \) is continuous. Similarly for \( p = 1 \).

If \( \phi \in \Lambda^* \) is given by \( \phi(\sum_{n=1}^{\infty} a_n z^n) = \sum_{n=1}^{\infty} a_n b_n \), then \( (T^* \phi)(a) = \phi(T(a)) = \sum_{n=1}^{\infty} a_n b_n \) for every \( a \in \ell_p \). So \( T^* \phi \) is the sequence \( \{b_n\} \) considered as a member of \( (\ell_p)^* = \ell_1 \). Thus \( \{b_n\} \) is in the range of \( T^* \) if and only if \( \limsup_n \frac{1}{|b_n|} < 1 \).

(1) Suppose \( f = Ta \in \text{Supp } P(p) \). By Proposition 3.2, \( a \in \text{Supp } \text{Ball}(\ell_p) \). Thus by Proposition 3.1, we get \( \sum_{n=1}^{\infty} |a_n|^p = 1 \). If the functional associated with \( Ta \) is given by \( \{b_n\}_{n=1}^{\infty} \), then \( \limsup_n \frac{1}{|b_n|} < 1 \). By Proposition 3.3, there exists \( \beta \neq 0 \) such that \( |a_n|^p = \beta |b_n|^q \) for all \( n \). Thus \( \limsup_n \frac{1}{|b_n|} < 1 \) and so \( f \) is analytic in \( \overline{D} \).

Conversely, suppose that \( f = T(a) \) is analytic in \( \overline{D} \) with \( \sum_{n=1}^{\infty} |a_n|^p = 1 \). Then \( a \in \text{Supp } \ell_p \). By Proposition 3.1, and one can choose the functional associated with \( a \) as in the formula of Proposition 3.3. Since the radius of convergence of the power series of \( f \) is greater than one, \( \limsup_n \frac{1}{|b_n|} < 1 \) and \( \limsup_n \frac{1}{|b_n|} < 1 \), and thus \( b \in \text{range } T^* \). Thus \( f \in \text{Supp } P(p) \) by Proposition 3.2.

(2) Suppose \( f = Ta \in \text{Supp } P(1) \) and \( b \) is a functional associated with \( a \). Then \( |a_n| \leq 1 \) and \( b(a) = \|b\|_{\infty} \) by Propositions 3.2 and 3.1. Thus equality must hold at all points of the chain \( b(a) = \sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} |a_n| |b|_{\infty} \leq |b|_{\infty} \). In particular \( |b_n| = |b|_{\infty} \) whenever \( a_n \neq 0 \). Since \( \limsup_n \frac{1}{|b_n|} < 1 \), this means \( a_n = 0 \) for all but finitely many \( n \), as required.

Conversely, suppose \( Ta = f(z) = \sum_{n=1}^{\infty} a_n z^n \) and \( \sum_{n=1}^{\infty} |a_n| = 1 \). Then \( a \in \text{Supp } \ell_1 \).

Define \( b_n = \begin{cases} \frac{a_n}{|a_n|} & \text{if } a_n \neq 0, \\ 0 & \text{otherwise}. \end{cases} \)

Then \( \limsup_n \frac{1}{|b_n|} < 1 \) and \( \{b_n\}_{n=1}^{\infty} \in (\ell_p)^* \) is associated with \( a \). By Proposition 3.2, \( f \) is a support point of \( P(1) \), as required.

Let \( Q(p) = \{f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A : \sum_{n=2}^{\infty} n|a_n|^p \leq 1 \}, 1 \leq p < \infty \). The class \( Q(1) \) has been studied in [6].

We remark that each element of \( Q(1) \) is univalent.
COROLLARY 3.5. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a support point of $Q(p)$ if and only if

1. $f$ is analytic in $\overline{D}$ and $\sum_{n=2}^{\infty} n|a_n|^p = 1$, if $1 < p < \infty$.

2. $f(z) = z + \sum_{n=2}^{N} a_n z^n$ and $\sum_{n=2}^{N} |a_n| = 1$, for some positive integer $N \geq 2$, if $p = 1$.

PROOF. One way to see this is to replace $\ell_p$ by $\ell_p(\mu)$, where $\mu(n) = n$, $n = 2, 3, \ldots$ in the proof of Theorem 3.4.

REMARK. One can define $P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : \sup|a_n| \leq 1\}$. One can show, using an argument similar to the proof of Theorem 3.4, that $\text{Supp } P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : |a_n| = 1$ for some $n \geq 1\}$.

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