ON GENERALIZATION OF CONTINUED FRACTION OF GAUSS

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ABSTRACT. In this paper we establish a continued fraction representation for the ratio of two basic bilateral hypergeometric series \( \psi \)'s which generalize Gauss' continued fraction for the ratio of two \( F_1 \)'s.

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1. INTRODUCTION.

Gauss (see Wall [3] and also Jones and Thron [2], gave the following continued fraction involving the ratio of two Gaussian \( F_1 \)'s,

\[
\frac{a,b+1,z}{c+1} / \frac{a,b,z}{c}
\]

where

\[
1 - \frac{(a+1)(b+1)z/(c+2)(c+3)}{1 - \frac{(a+1)(c-b)z/(c+3)(c+4)}{1 - \frac{(c+2)(c+3) - 1 - (b+2)(c-a+2)z/(c+3)(c+4)}}}
\]

In this paper we establish the continued fraction for the ratio

\[
\psi_2 \left[ \begin{array}{c} \alpha,\beta \end{array} \right]_{\delta,\gamma} / \psi_2 \left[ \begin{array}{c} \alpha,\beta \end{array} \right]_{\delta,\gamma q}
\]

where

\[
\psi_2 \left[ \begin{array}{c} \alpha,\beta \end{array} \right]_{\delta,\gamma} = \sum_{n=0}^\infty \frac{[a]_n [\beta]_n z^n}{[\delta]_n [\gamma]_n} , \quad (|z| < 1)
\]

in which the symbol \([a]_n\) stands for \(a(a+1)(a+2)...(a+n-1)\) and \([a]_0 = 1\).

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\]

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\[
\psi_2 \left[ \begin{array}{c} \alpha,\beta \end{array} \right]_{\delta,\gamma} = \sum_{n=0}^\infty \frac{[a]_n [\beta]_n x^n}{[\delta]_n [\gamma]_n} , \quad (|\delta\gamma/|\alpha\beta| < |x| < 1, \quad |q| < 1)
\]
where

\[ [\alpha]_n \equiv [\alpha; q]_n = (1 - \alpha)(1 - \alpha q) \cdots (1 - \alpha q^{n-1}), [\alpha]_0 = 1. \]

The other notations appearing in this paper carry their usual meaning.

2. MAIN RESULT.

In this paper we establish the following result

\[ 2\psi_2 \left[ \frac{\alpha, \beta q; x}{\delta, \gamma q} \right] / 2\psi_2 \left[ \frac{\alpha, \beta; x}{\delta, \gamma} \right] \]

\[ = \frac{1}{A_0 + C_0 + A_1 + C_1 + A_2 + C_2 + \cdots}, \]

where for \( i = 0, 1, 2, 3, \ldots \)

\[ A_i = \frac{\gamma q^{2i+1} - \delta}{(1 - \gamma q^{2i+1}) (\beta q^{i+1} - \delta)}, \]

\[ B_i = \frac{q^{i+1} (1 - \alpha q^i) (1 - \beta q^i) (\gamma q^i - \delta)}{(1 - \gamma q^{2i+1}) (1 - \gamma q^i) (\beta q^{i+1} - \delta)}, \]

\[ C_i = \frac{(1 - \alpha q^i) (\gamma q^{2i+2} - \delta)}{(1 - \gamma q^{2i+1}) (\alpha q^{i+1} - \delta)}, \]

and

\[ D_i = \frac{q^{i+1} (1 - \beta q^{i+1}) (1 - \alpha q^i) (\gamma q^{i+1} - \delta)}{(1 - \gamma q^{2i+1}) (1 - \gamma q^{2i+2}) (\alpha q^{i+1} - \delta)}. \]

PROOF of (2.1). It is easy to see that the following relation is true (for non-negative integral \( i \)),

\[ 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+1}}{\delta, \gamma q^{2i+1}} \right] \]

\[ = A_i 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+1}}{\delta, \gamma q^{2i+1}} \right] + B_i 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+2}}{\delta, \gamma q^{2i+2}} \right] \]

\[ = C_i 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+1}}{\delta, \gamma q^{2i+1}} \right] + D_i 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+2}}{\delta, \gamma q^{2i+2}} \right] \]

\[ + x \delta \psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+2}}{\delta, \gamma q^{2i+2}} \right] \]

\[ = C_1 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+1}}{\delta, \gamma q^{2i+1}} \right] + x D_1 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+2}}{\delta, \gamma q^{2i+2}} \right] \]

\[ + x D_2 2\psi_2 \left[ \frac{\alpha^i q; \delta, \gamma q^{2i+2}}{\delta, \gamma q^{2i+2}} \right] \]

\[ \cdots \]
Now from (2.2) for \( i = 0 \), we get

\[
2^2_2 \left[ \frac{a, \beta; x}{\delta, \gamma_q} \right] / 2^2_2 \left[ \frac{a, \beta q; x}{\delta, \gamma_q^2} \right]
\]

\[
= A_0 + \frac{xB_0}{2^2_2 \left[ \frac{a, \beta q; x}{\delta, \gamma_q^2} \right] / 2^2_2 \left[ \frac{a, \beta q; x}{\delta, \gamma_q^2} \right]}
\]

\[
= A_0 + \frac{xB_0}{2^2_2 \left[ \frac{a, \beta q; x}{\delta, \gamma_q^2} \right] / 2^2_2 \left[ \frac{a, \beta q^2; x}{\delta, \gamma_q^4} \right]}
\]

from (2.3) with \( i = 0 \)

\[
= A_0 + \frac{xB_0}{C_0 + \frac{xD_0}{xB_1}}
\]

from (2.2) with \( i = 1 \)

\[
= A_0 + \frac{xB_0}{C_0 + \frac{xD_0}{xB_1}} + \frac{xD_1}{xB_2} + \ldots
\]

(by repeated application of (2.2) and (2.3)). This proves (2.1).

3. SPECIAL CASES.

Here we shall reduce certain interesting special cases of (2.1). If in (2.1) we take \( \delta = q \), we get

\[
2^2_1 \left[ \frac{a, \beta q; x}{\gamma_q} \right] / 2^1_1 \left[ \frac{a, \beta; x}{\gamma} \right]
\]

\[
= \frac{1}{1} + \frac{x\mu_0}{1} + \frac{x\nu_0}{1} + \frac{x\eta_1}{1} + \frac{x\nu_1}{1} + \frac{x\mu_2}{1} + \ldots
\]

where for \( i = 0, 1, 2, \ldots \)

\[
\mu_i = q^i (1-aq^i)(\gamma_q^i-\beta)/(1-\gamma_q^{2i})(1-\gamma_q^{2i+1})
\]

and

\[
\nu_i = q^i (1-\beta q^{i+1})(\gamma_q^{i+1}-\alpha)/(1-\gamma_q^{2i+1})(1-\gamma_q^{2i+2})
\]

If \( q > 1 \) in (3.1), we get (1.1), the continued fraction of Gauss.

If in (3.1) we take \( \beta = 1 \) and replace \( \gamma \) by \( \gamma/q \), we get,
\[2F_2 \left[ \begin{array}{c} a, q; \gamma \end{array} \middle| x \right] = \frac{1}{1 + \frac{x\mu_0}{1 + \frac{x\nu_0}{1 + \frac{x\mu_1}{1 + \frac{x\nu_1}{1 + \frac{x\mu_2}{1 + \cdots}}}}} \tag{3.2}\]

where for \( i = 0, 1, 2, \ldots \)

\[\mu_i = -q^i(1-aq^i)(1-\gamma q^{i-1})(1-\gamma q^{2i-1})(1-\gamma q^{2i})\]

and

\[\nu_i = -aq^i(1-\gamma q^i)(1-\gamma q^{i-1})(1-\gamma q^{2i})(1-\gamma q^{2i+1})\].

Now, if in (3.2) we let \( q = 1 \), we get the following known result [2]

\[F \left[ \begin{array}{c} a, 1; x \end{array} \middle| \gamma \right] = \frac{1}{1 - \frac{x\xi_0}{1 - \frac{x\eta_0}{1 - \frac{x\xi_1}{1 - \frac{x\eta_1}{1 - \frac{x\xi_2}{1 - \cdots}}}}} \tag{3.3}\]

where for \( i = 0, 1, 2, \ldots \)

\[\xi_i = (a+i)(\gamma+i-1)/(\gamma+2i-1)(\gamma+2i)\]

and

\[\eta_i = (i+1)(\gamma-a+i)/(\gamma+2i)(\gamma+2i+1)\].

If we put \( \gamma = 0 \) in (3.2) and replace \( x \) by \( xq/\alpha \) and then let \( \alpha \rightarrow \infty \), we get the following interesting result

\[\frac{\Gamma}{n} \frac{(-)^n}{q \Gamma(n+1)/2} x^n \]

\[= \frac{1}{1 + \frac{xq}{1 + \frac{xq(q-1)}{1 + \frac{xq^2(q-1)}{1 + \frac{xq^3(q-1)}{1 + \cdots}}}}} \tag{3.4}\]

If we take \( \gamma = q \) in (3.2) we get a continued fraction representation for \( \phi_1 [a; -; x] \) which, when \( q = 1 \), yields the continued fraction representation for general binomial \((1-x)^{-\alpha}\).

Again, if we take \( \alpha = q \), \( \gamma = q^2 \) and replace \( x \) by \(-x\) in (3.2), we get a continued fraction representation for \( 2F_1 [q, q; q^2; -x] \) which, when \( q = 1 \) yields the continued fraction representation for

\[\frac{1}{x-\log(1+x)} = F \left[ \begin{array}{c} 1,1; -x \end{array} \middle| 2 \right] \]

Similarly, we can get the continued fraction representation for

\[\log \left( \frac{1+x}{1-x} \right) = 2x F \left[ \begin{array}{c} 1/2, 1; x \end{array} \middle| 3/2 \right] \]
Further, if we take $\alpha = 0$ in (3.1), we get the following result after some simplification,

$$1 \ell_1 \left[ \frac{\beta; x}{\gamma} \right] = 1 + \frac{x_{0}}{1} + \frac{x_{1}}{1 + \frac{x_{0}}{1}} + \frac{x_{2}}{1 + \frac{x_{1}}{1 + \frac{x_{0}}{1}}} + \cdots ,$$

(3.5)

where for $i = 0, 1, 2, \ldots$

$$u_{i} = q^{i} (\gamma_{i} q - 1) / (1 - \gamma_{i} q) (1 - \gamma_{i+1} q)$$

and

$$v_{i} = \gamma_{i+1} q (1 - \gamma_{i+1}) / (1 - \gamma_{i+1} q) (1 - \gamma_{i+2}) .$$

The above (3.5) is the $q$-analogue of a known result [2].

Again, setting $\beta = 1$ in (3.5) we get the continued fraction representation for $1 \ell_1 \left[ \frac{q^{1} x}{\gamma} \right]$ from which one can, for $\gamma = 1$, deduce the corresponding continued fraction expression for $q$-exponential function $eq(x)$ which in turn yields the continued fraction representation for exponential function $e^z$ when $q + 1 = 1$ [2].

A number of other interesting special cases could also be deduced. The reader is referred to Wall [1] and Jones [2].

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