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ON GENERALIZATION OF CONTINUED FRACTION OF GAUSS

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ABSTRACT. In this paper we establish a continued fraction represetation for the ratio qf two basic bilateral hypergeometric series ${}_2\psi_2$'s which generalize Gauss' continued fraction for the ratio of two ${}_2F_1$'s.

KEY WORDS AND PHRASES. Continued fractions and hypergeometric series.
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1. INTRODUCTION.

Gauss (see Wall [3] and also Jones and Thron [2], gave the following continued fraction involving the ratio of two Gaussian ${}_2F_1$'s,

$$F\left[\begin{matrix}a,b+1;z\\c+1\end{matrix}\right]/F\left[\begin{matrix}a,b;z\\c\end{matrix}\right] \tag{1.1}$$

$$= \cfrac{1}{1-\cfrac{a(c-b)z/c(c+1)}{1-\cfrac{(b+1)(c-a+1)z/(c+1)(c+2)}{1-\cfrac{(a+1)(c-b+1)z/(c+2)(c+3)}{1-\cfrac{(b+2)(c-a+2)z/(c+3)(c+4)}{1-\quad\vdots}}}}}$$

where

$${}_2F_1\left[\begin{matrix}\alpha,\beta;z\\\gamma\end{matrix}\right]=\sum_{n=o}^{\infty}\frac{[\alpha]_n[\beta]_n z^n}{[1]_n[\gamma]_n},\ (|z|<1)$$

in which the symbol $[\alpha]_n$ stands for $\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)$ and $[\alpha]_o = 1$.

In this paper we establish the continued fraction for the ratio

$${}_2\psi_2\left[\begin{matrix}\alpha,\beta;x\\\delta,\gamma\end{matrix}\right]\Big/\ {}_2\psi_2\left[\begin{matrix}\alpha,\beta q;x\\\delta,\gamma q\end{matrix}\right]$$

where

$${}_2\psi_2\left[\begin{matrix}\alpha,\beta;x\\\delta,\gamma\end{matrix}\right]=\sum_{n=-\infty}^{\infty}\frac{[\alpha]_n[\beta]_n x^n}{[\delta]_n[\gamma]_n},\ (|\delta\gamma/\alpha\beta|<|x|<1,\ |q|<1),$$

where

$$[\alpha]_n \equiv [\alpha;q]_n = (1-\alpha)(1-\alpha q)\ldots(1-\alpha q^{n-1}), [\alpha]_o = 1.$$

The other notations appearing in this paper carry their usual meaning.

2. MAIN RESULT.

In this paper we establish the following result

$$ {}_2\psi_2 \left[\begin{matrix}\alpha,\beta q;x\\ \delta,\gamma q\end{matrix}\right] \Big/ {}_2\psi_2 \left[\begin{matrix}\alpha,\beta;x\\ \delta,\gamma\end{matrix}\right] \tag{2.1}$$

$$= \frac{1}{A_o} + \frac{xB_o}{C_o} + \frac{xD_o}{A_1} + \frac{xB_1}{C_1} + \frac{xD_1}{A_2} + \frac{xB_2}{C_2} + \cdots\cdots,$$

where for $i = 0, 1, 2, 3, \ldots$

$$A_i = \frac{(1-\beta q^i)(\gamma q^{2i+1} - \delta)}{(1-\gamma q^{2i})(\beta q^{i+1} - \delta)},$$

$$B_i = \frac{q^{i+1}(1-\alpha q^i)(1-\beta q^i)(\beta-\gamma q^i)}{(1-\gamma q^{2i+1})(1-\gamma q^{2i})(\beta q^{i+1} - \delta)},$$

$$C_i = \frac{(1-\alpha q^i)(\gamma q^{2i+2} - \delta)}{(1-\gamma q^{2i+1})(\alpha q^{i+1} - \delta)}$$

and

$$D_i = \frac{q^{i+1}(1-\beta q^{i+1})(1-\alpha q^i)(\alpha-\gamma q^{i+1})}{(1-\gamma q^{2i+1})(1-\gamma q^{2i+2})(\alpha q^{i+1} - \delta)}.$$

PROOF of (2.1). It is easy to see that the following relation is true (for non-negative integral i),

$$ {}_2\psi_2 \left[\begin{matrix}\alpha q^i,\beta q^i;x\\ \delta,\ \gamma q^{2i}\end{matrix}\right]$$

$$= A_i\ {}_2\psi_2 \left[\begin{matrix}\alpha q^i,\beta q^{i+1};x\\ \delta,\ \gamma q^{2i+1}\end{matrix}\right] + xB_i\ {}_2\psi_2 \left[\begin{matrix}\alpha q^{i+1},\ \beta q^{i+1};\ x\\ \delta\ ,\ \gamma q^{2i+2}\end{matrix}\right] \tag{2.2}$$

Now, interchanging α and β in (2.2) and then replacing β by βq and γ by γq in it, we get

$$ {}_2\psi_2 \left[\begin{matrix}\alpha q^i,\ \beta q^{i+1};x\\ \delta,\ \gamma q^{2i+1}\end{matrix}\right]$$

$$= C_i\ {}_2\psi_2 \left[\begin{matrix}\alpha q^{i+1},\ \beta q^{i+1};x\\ \delta\ ,\ \gamma q^{2i+2}\end{matrix}\right] + xD_i\ {}_2\psi_2 \left[\begin{matrix}\alpha q^{i+1},\ \beta q^{i+2};x\\ \delta\ ,\ \gamma q^{2i+3}\end{matrix}\right] \tag{2.3}$$

Now from (2.2) for i = o, we get

$$ {}_2\psi_2\left[\begin{matrix}\alpha,\beta;x\\ \delta,\gamma\end{matrix}\right] \Big/ {}_2\psi_2\left[\begin{matrix}\alpha,\beta q;x\\ \delta,\ \gamma q\end{matrix}\right] $$

$$ = A_o + \frac{xB_o}{{}_2\psi_2\left[\begin{matrix}\alpha,\beta q;x\\ \delta,\gamma q\end{matrix}\right] \Big/ {}_2\psi_2\left[\begin{matrix}\alpha q,\ \beta q;x\\ \delta,\ \gamma q^2\end{matrix}\right]} $$

$$ = A_o + \cfrac{xB_o}{C_o + \cfrac{xD_o}{{}_2\psi_2\left[\begin{matrix}\alpha q,\beta q;x\\ \delta,\ \gamma q^2\end{matrix}\right] \Big/ {}_2\psi_2\left[\begin{matrix}\alpha q,\beta q^2;x\\ \delta\ ,\ \gamma q^3\end{matrix}\right]}} $$

from (2.3) with i = o

$$ = A_o + \cfrac{XB_o}{C_o + \cfrac{xD_o}{A_1 + \cfrac{xB_1}{{}_2\psi_2\left[\begin{matrix}\alpha q,\beta q^2;x\\ \delta\ ,\ \gamma q^3\end{matrix}\right] \Big/ {}_2\psi_2\left[\begin{matrix}\alpha q^2,\ \beta q^2,\ x\\ \delta,\ \gamma q^4\end{matrix}\right]}}} $$

from (2.2) with i = 1

$$ = A_o + \frac{xB_o}{C_o} + \frac{xD_o}{A_1} + \frac{xB_1}{C_1} + \frac{xD_1}{A_2} + \frac{xB_2}{C_2} + \cdots $$

(by repeated application of (2.2) and (2.3)). This proves (2.1).

3. SPECIAL CASES.

Here we shall reduce certain interesting special cases of (2.1).If in (2.1) we take $\delta = q$, we get

$$ {}_2\phi_1\left[\begin{matrix}\alpha,\beta q;x\\ \gamma q\end{matrix}\right] \Big/ {}_2\phi_1\left[\begin{matrix}\alpha,\ \beta;\ x\\ \gamma\end{matrix}\right] $$

$$ = \frac{1}{1+} + \frac{x\mu_o}{1} + \frac{x\nu_o}{1} + \frac{x\eta_1}{1} + \frac{x\nu_1}{1} + \frac{x\mu_2}{1} + \cdots, \tag{3.1} $$

where for i = 0, 1, 2, . . .

$$ \mu_i = q^i\ (1-\alpha q^i)(\gamma q^i-\beta)/(1-\gamma q^{2i})(1-\gamma q^{2i+1}) $$

and

$$ \nu_i = q^i\ (1-\beta q^{i+1})(\gamma q^{i+1}-\alpha)/(1-\gamma q^{2i+1})(1-\gamma q^{2i+2}). $$

If $q \to 1$ in (3.1), we get (1.1), the continued fraction of Gauss.

If in (3.1) we take $\beta = 1$ and replace γ by γ/q, we get,

$$ {}_2\Phi_2\left[\begin{matrix}\alpha,\ q;\ x\\ \gamma\end{matrix}\right] $$

$$ = \frac{1}{1} + \frac{x\mu_o}{1} + \frac{x\nu_o}{1} + \frac{x\mu_1}{1} + \frac{x\nu_1}{1} + \frac{x\mu_2}{1} + \cdots, \tag{3.2} $$

where for $i = 0, 1, 2, \ldots$

$$ \mu_i = -q^i(1-\alpha q^i)(1-\gamma q^{i-1})/(1-\gamma q^{2i-1})(1-\gamma q^{2i}) $$

and

$$ \nu_i = -\alpha q^i\,(1-q^{i+1})(1-\gamma q^i/\alpha)/(1-\gamma q^{2i})(1-\gamma q^{2i+1}) . $$

Now, if in (3.2) we let $q \to 1$, we get the following known result [2]

$$ F\left[\begin{matrix}\alpha,\ 1;\ x\\ \gamma\end{matrix}\right] $$

$$ = \frac{1}{1} - \frac{x\xi_o}{1} - \frac{x\eta_o}{1} - \frac{x\xi_1}{1} - \frac{x\eta_1}{1} - \frac{x\xi_2}{1} - \cdots, \tag{3.3} $$

where for $i = 0, 1, 2, \ldots$

$$ \xi_i = (\alpha+i)(\gamma+i-1)/(\gamma+2i-1)(\gamma+2i) $$

and

$$ \eta_i = (i+1)(\gamma-\alpha+i)/(\gamma+2i)(\gamma+2i+1) . $$

If we put $\gamma = o$ in (3.2) and replace x by xq/α and then let $\alpha \to \infty$, we get the following interesting result

$$ \sum_{n=o}^{\infty} (-)^n q^{n(n+1)/2} x^n $$

$$ = \frac{1}{1} + \frac{xq}{1} + \frac{xq(q-1)}{1} + \frac{xq^3}{1} + \frac{xq^2(q^2-1)}{1} + \frac{zq^5}{1} + \frac{xq^3(q^3-1)}{1} + \cdots, \tag{3.4} $$

If we take $\gamma = q$ in (3.2) we get a continued fraction representation for ${}_1\Phi_o\ [\alpha;\ -\ ;\ x]$ which, when $q \to 1$, yields the continued fraction representation for general binomial $(1-x)^{-\alpha}$.

Again, if we take $\alpha = q$, $\gamma = q^2$ and replace x by $-x$ in (3.2), we get a continued fraction representation for ${}_2\Phi_1\ [q,q;q^2;-x]$ which, when $q \to 1$ yields the continued fraction representation for

$$ \frac{1}{x}\log\,(1+x) = F\left[\begin{matrix}1,1;\ -x\\ 2\end{matrix}\right] . $$

Similarly, we can get the continued fraction representation for

$$ \log\,\left(\frac{1+x}{1-x}\right) = 2x\ F\left[\begin{matrix}1/2,\ 1;x\\ 3/2\end{matrix}\right] . $$

Further, if we take $\alpha = o$ in (3.1), we get the following result after some simplification,

$$ {}_1\Phi_1\left[\begin{matrix}\beta;x\\ \gamma\end{matrix}\right] \Big/ {}_1\Phi_1\left[\begin{matrix}\beta q;x\\ \gamma q\end{matrix}\right] $$

$$ = 1 + \frac{x\mu_o}{1} + \frac{x\nu_o}{1} + \frac{x\mu_1}{1} + \frac{x\nu_1}{1} + \frac{x\mu_2}{1} + \cdots , \tag{3.5} $$

where for $i = 0, 1, 2, \ldots$

$$ \mu_i = q^i(\gamma q^i-\beta)/(1-\gamma q^{2i})(1-\gamma q^{2i+1}) $$

and

$$ \nu_i = \gamma q^{2i+1}(1-\beta q^{i+1})/(1-\gamma q^{2i+1})(1-\gamma q^{2i+2}) \quad . $$

The above (3.5) is the q-analogue of a known result [2].

Again, setting $\beta = 1$ in (3.5) we get the continued fraction representation for ${}_1\Phi_1[{}^{q;x}_{\gamma q}]$ from which one can, for $\gamma=1$, deduce the corrsponding continued fraction expression for q-exponential function eq(x) which in turn yields the continued fraction representation for exponential function e^z when $q \to 1$ [2].

A number of other interesting special cases could also be deduced. The reader is referred to Wall [1] and Jones [2].

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