ON CLOSE-TO-CONVEX FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. The class $S^*(b)$ of starlike functions of complex order b was introduced and studied by M.K. Aouf and M.A. Nasr. The authors using the Ruscheweyh derivatives introduce the class $K(b)$ of functions close-to-convex of complex order b , $b \neq 0$ and its generalization, the classes $K_n(b)$ where n is a nonnegative integer. Here $S^*(b) \subset K(b) = K_0(b)$. Sharp coefficient bounds are determined for $K_n(b)$ as well as several sufficient conditions for functions to belong to $K_n(b)$. The authors also obtain some distortion and covering theorems for $K_n(b)$ and determine the radius of the largest disk in which every $f \in K_n(b)$ belongs to $K_n(1)$. All results are sharp.

KEY WORDS AND PHRASES. Starlike functions, close-to-convex functions of complex order, Ruscheweyh derivatives, Hadamard product.

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1. INTRODUCTION.

Let A denote the class of functions $f(z)$ analytic in the unit disk $E = \{z: |z| < 1\}$ having the power series

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad z \in E. \quad (1.1)$$

Aouf and Nasr [1] introduced the class $S^*(b)$ of starlike functions of order b , where b is a nonzero complex number, as follows:

$$S^*(b) = \{f: f \in A \text{ and } \operatorname{Re} \left[1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right] > 0, \quad z \in E\}.$$

We define the class $K(b)$ of close-to-convex functions of complex order b as follows: $f \in K(b)$ if and only if $f \in A$ and

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left(\frac{zf'(z)}{g(z)} - 1 \right) \right\} > 0, \quad z \in E, \quad (1.2)$$

for some starlike function g .

The classes R_n , $n \in N_0$ and where N_0 is the set of nonnegative integers, were introduced by Singh and Singh [2], $f \in R_n$ if and only if $f \in A$ and

$$\operatorname{Re} \frac{z(D^n f(z))'}{D^n f(z)} > 0, \quad z \in E, \quad (1.3)$$

where

$$D^n f(z) = f(z) * \frac{z}{(1-z)^{n+1}}, \quad (1.4)$$

and $(*)$ stands for the Hadamard product of power series, i.e., if

$$f(z) = \sum_0^{\infty} a_n z^n, \quad g(z) = \sum_0^{\infty} b_n z^n \text{ then } f(z) * g(z) = \sum_0^{\infty} a_n b_n z^n.$$

The operator D^n is referred to in Al-Amiri [3] as the Ruscheweyh derivative of order n . Note that R_0 is the familiar class of starlike functions, S^* . More, it is known [2] that $R_{n+1} \subset R_n$, $n \in N_0$, and consequently R_n consists of functions starlike in E .

Let $K_n(b)$, $n \in N_0$, b is a nonzero complex number, denote the class of functions $f \in A$ satisfying

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} \left[\frac{z(D^n f(z))'}{D^n g(z)} - 1 \right] \right\} > 0, \quad z \in E, \quad (1.5)$$

for some $g \in R_n$. Here $K_0(b) = K(b)$.

Many authors have studied various classes of univalent and multivalent functions using the Ruscheweyh derivatives D^n , $n \in N_0$. In particular one can look at the work of Ruscheweyh [4].

Section 2 determines coefficient estimates of functions in $K_n(b)$, $n \in N_0$. In section 3, we obtain some distortion and covering theorems for $K_n(b)$ and several sufficient conditions for functions to be in $K_n(b)$. The radius of close-to-convexity for the class of close-to-convex of complex order b is also determined in section 3.

2. COEFFICIENT ESTIMATES.

In this section, sharp estimates for the coefficients of functions in $K_n(b)$ are determined in Theorem 2.1. First, we need the following lemmas.

LEMMA 2.1. For $n \in N_0$, let

$$(D^n f(z))' = \frac{1 + (2b-1)z}{(1-z)^3}. \quad (2.1)$$

Then $f \in K_n(b)$.

PROOF. Let $g \in A$ be defined so that

$$D^n g(z) = \frac{z}{(1-z)^2}. \quad (2.2)$$

The definition of R_n implies $g \in R_n$. A brief computation gives

$$1 + \frac{1}{b} \left[\frac{z(D^n f(z))'}{D^n g(z)} - 1 \right] = \frac{1+z}{1-z}, \quad z \in E.$$

This proves that $f \in K_n(b)$.

REMARK 2.1. The function f as defined in (2.1) has the power series representation in E

$$f(z) = z + \sum_{m=2}^{\infty} \frac{n! (M-1)!}{(n+m-1)!} [(m-1)b + 1] z^m. \quad (2.3)$$

LEMMA 2.2. Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m \in R_n$ where $n \in N_0$.

Then $|c_m| < \frac{n! m!}{(n+m-1)!}$.

PROOF. A brief computation gives

$$D^n g(z) = z + \sum_{m=2}^{\infty} \frac{(n+m-1)!}{n! (m-1)!} c_m z^m.$$

Since $g \in R_n$, $D^n g(z) \in S^*$. Thus, using the well known coefficient estimates for starlike functions one gets

$$\frac{(n+m-1)!}{n! (m-1)!} |c_m| < m, \quad m > 2,$$

and the proof is complete.

LEMMA 2.3. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in K_n(b)$, $n \in N_0$, then

$$\begin{aligned} |ma_m - c_m|^2 &< 4 \left[\frac{(m-1)!}{(n+m-1)!} \right]^2 |b| \\ &\cdot \left\{ n!^2 |b| + \sum_{k=2}^{m-1} \left[\frac{(n+k-1)!}{(k-1)!} \right]^2 \left[|ka_k - c_k| |c_k| + |b| |c_k|^2 \right] \right\} \quad (2.4) \end{aligned}$$

PROOF. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $K_n(b)$. Then (1.5) implies

$$1 + \frac{1}{b} \left[\frac{z(D^n f(z))'}{D^n g(z)} - 1 \right] = \frac{1+w(z)}{1-w(z)}, \quad z \in E, \quad (2.5)$$

for some $g \in R_n$ and where $w \in A$ such that $w(0) = 0$, $w(z) \neq 1$ and $|w(z)| < 1$ for

$z \in E$. Let $g(z) = z + \sum_{m=2}^{\infty} c_m z^m$. Then (2.5) and the Definition 1.4 imply

$$\begin{aligned} w(z) \{n! 2bz + \sum_{k=2}^{\infty} \frac{(n+k-1)!}{(k-1)!} [ka_k + (2b-1)c_k] z^k\} \\ = \sum_{k=2}^{\infty} \frac{(n+k-1)!}{(k-1)!} (ka_k - c_k) z^k. \end{aligned} \quad (2.6)$$

Using Clunie's method, that is to examine the bracketed quantity of the left-hand side in (2.6) and keep only those terms that involve z^k for $k < m-1$ for some fixed m , moving the other terms to the right side, one obtains

$$\begin{aligned} w(z) \{n! 2bz + \sum_{k=2}^{m-1} \frac{(n+k-1)!}{(k-1)!} [ka_k + (2b-1)c_k] z^k\} \\ = \sum_{k=2}^m \frac{(n+k-1)!}{(k-1)!} (ka_k - c_k) z^k + \sum_{k=m+1}^{\infty} A_k z^k. \end{aligned}$$

Let

$$\begin{aligned} \varphi(z) = w(z) \{n! 2bz + \sum_{k=2}^{m-1} \frac{(n+k-1)!}{(k-1)!} [ka_k + (2b-1)c_k] z^k\} \\ = \sum_{k=2}^m \frac{(n+k-1)!}{(k-1)!} (ka_k - c_k) z^k + \sum_{k=m+1}^{\infty} A_k z^k. \end{aligned} \quad (2.7)$$

Let $z = re^{i\theta}$, $0 < r < 1$. Computing $\frac{1}{2\pi} \int_0^{2\pi} \varphi(z) \overline{\varphi(z)} dz$ for both expressions of $\varphi(z)$ in (2.7) and using $|w(z)| < 1$ we get

$$\begin{aligned} \sum_{k=2}^m \left[\frac{(n+k-1)!}{(k-1)!} \right]^2 |ka_k - c_k|^2 r^{2k} \\ < n!^2 4|b|^2 r^2 + \sum_{k=2}^{m-1} \left[\frac{(n+k-1)!}{(k-1)!} \right]^2 |ka_k + (2b-1)c_k|^2 r^{2k}. \end{aligned}$$

Upon letting $r \rightarrow 1^-$ and after some easy computations we obtain

$$\begin{aligned} |ma_m - c_m|^2 < \left[\frac{(m-1)!}{(n+m-1)!} \right]^2 4|b| \left\{ n!^2 |b| + \sum_{k=2}^{m-1} \left[\frac{(n+k-1)!}{(k-1)!} \right]^2 \right. \\ \left. \cdot [|ka_k - c_k| |c_k| + |b| |c_k|^2] \right\}. \end{aligned}$$

In particular, when $m = 2$ we have

$$|2a_2 - c_2| < \frac{1}{n+1} 2|b|. \quad (2.8)$$

The proof of the lemma is complete.

THEOREM 2.1. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. If $f \in K_n(b)$ where $n \in N_0$,

then

$$|a_m| < \frac{n! (m-1)!}{(n+m-1)!} [(m-1)|b| + 1].$$

This result is sharp. An extremal function is given by (2.3).

PROOF. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ be in $K_n(b)$. Let the associate function of f ,

$g(z) = z + \sum_{m=2}^{\infty} c_m z^m$. We claim that for $m \geq 2$ and $n \in N_0$,

$$|ma_m - c_m| < \frac{n! (m-1)!}{(n+m-1)!} 2|b| \left[1 + \sum_{k=2}^{m-1} \frac{(n+k-1)!}{n! (k-1)!} |c_k| \right]. \quad (2.9)$$

We use the second principle of finite induction on m to prove (2.9).

For $m = 2$, $|2a_2 - c_2| < \frac{n!}{(n+1)!} \cdot 2|b| = \frac{2(b)}{(n+1)}$ is true as shown in (2.8). Now assume (2.9) is true for all $m < p$. Taking $m = p+1$ in (2.4), we get

$$\begin{aligned} |(p+1)a_{p+1} - c_{p+1}|^2 &< 4 \left[\frac{p!}{(n+p)!} \right]^2 |b| \\ &\cdot \left\{ n!^2 |b| + \left[\sum_{k=2}^p \frac{(n+k-1)!}{(k-1)!} \right]^2 |ka_k - c_k| |c_k| + |b| |c_k|^2 \right\} \\ &= 4 \left[\frac{p!}{(n+p)!} \right]^2 |b| \left\{ n!^2 |b| + \sum_{k=2}^p \left[\frac{(n+k-1)!}{(k-1)!} \right]^2 |ka_k - c_k| |c_k| \right. \\ &\quad \left. + |b| \sum_{k=2}^p \left[\frac{(n+k-1)!}{(k-1)!} \right]^2 |c_k|^2 \right\}. \end{aligned}$$

Now using (2.9) since $k < p$, the above yields

$$\begin{aligned} |(p+1)a_{p+1} - c_{p+1}|^2 &< 4 \left[\frac{n! p!}{(n+p)!} \right]^2 |b|^2 \left\{ 1 + 2 \sum_{k=2}^p \frac{(n+k-1)!}{n! (k-1)!} |c_k| \right. \\ &\cdot \left[1 + \sum_{\ell=2}^{k-1} \frac{(n+\ell-1)!}{n! (\ell-1)!} |c_\ell| \right] + \sum_{k=2}^p \left[\frac{(n+k-1)!}{n! (k-1)!} \right]^2 |c_k|^2 \Big\} \\ &= 4 \left[\frac{n! p!}{(n+p)!} \right]^2 |b|^2 \left\{ 1 + 2 \sum_{k=2}^p \frac{(n+k-1)!}{n! (k-1)!} |c_k| \right. \\ &\quad + 2 \sum_{k=2}^p \frac{(n+k-1)!}{n! (k-1)!} \left[|c_k| \sum_{\ell=2}^{k-1} \frac{(n+\ell-1)!}{n! (\ell-1)!} |c_\ell| \right] \\ &\quad \left. + \sum_{k=2}^p \left[\frac{(n+k-1)!}{n! (k-1)!} \right]^2 |c_k|^2 \right\}. \end{aligned}$$

Applying the principle of mathematical induction on p , it is easily seen that the sum of the last two terms appearing in the bracketed expression in the right hand side

of the above is equal to $\left[\sum_{k=2}^p \frac{(n+k-1)!}{n! (k-1)!} |c_k| \right]^2$. Consequently

it follows that

$$|(p+1)a_{p+1} - c_{p+1}|^2 < 4 \left[\frac{n! p!}{(n+p)!} \right]^2 |b|^2 \left[1 + \sum_{k=2}^p \frac{(n+k-1)!}{n! (k-1)!} |c_k| \right]^2.$$

This shows that (2.9) is valid for $m = p + 1$. Hence, by the second principle of finite induction, the claim is correct. From Lemma 2.2 and 2.9 it follows that

$$|ma_m - c_m| < \frac{n! m! (m-1)}{(n+m-1)!} |b|, \quad m > 2. \quad (2.10)$$

Finally from Lemma 2.2 and 2.10 we deduce that

$$|a_m| < \frac{n! (m-1)!}{(n+m-1)!} [(m-1)|b| + 1], \quad m > 2.$$

Hence the proof of the Theorem 2.1 is complete.

Putting $n = 0$ in Theorem 2.1 we have the following corollary.

COROLLARY 2.1. If $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$ is a close-to-convex function of complex

order b , then $|a_m| < (m-1)|b| + 1$. This result is sharp.

REMARK 2.2. For $b = 1$, Corollary 2.1 is reduced to the well known coefficient bounds for the close-to-convex functions due to Reade [5].

Next we have two theorems that provide sufficient conditions for a function to be in $K_n(b)$.

THEOREM 2.2. Let $f \in A$ and $n \in N_0$. If any of the following conditions is satisfied in E , then $f \in K_n(b)$.

$$(i) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} [(D^n f(z))' - 1] \right\} > 0,$$

$$(ii) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} [(1-z)(D^n f(z))' - 1] \right\} > 0,$$

$$(iii) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} [(1-z)^2 (D^n f(z))' - 1] \right\} > 0,$$

$$(iv) \quad \operatorname{Re} \left\{ 1 + \frac{1}{b} [(1-z)^2 (D^n f(z))' - 1] \right\} > 0.$$

PROOF. The proofs follow by choosing g as below:

$$(i) \quad g(z) = z,$$

$$(ii) \quad g(z) = z + \sum_{m=2}^{\infty} \frac{n! (m-1)!}{(n+m-1)!} z^m,$$

$$(iii) \quad g(z) = z + \sum_{m=2}^{\infty} \frac{n! (2m-2)!}{(n+2m-2)!} z^{2m-1}, \text{ and}$$

$$(iv) \quad g(z) = z + \sum_{m=2}^{\infty} \frac{n! m!}{(n+m-1)!} z^m \text{ respectively.}$$

THEOREM 2.3. Let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m$. For $n \in N_0$, each of the following

conditions is sufficient for f to be in $K_n(b)$.

$$(i) \quad \sum_{m=2}^{\infty} \frac{(n+m-1)!}{n! (m-1)!} m |a_m| < |b|.$$

$$(ii) \quad \sum_{m=2}^{\infty} \frac{(n+m-1)!}{n! (m-1)!} \left| m a_m - \frac{(n+m)(m+1)}{m} a_{m+1} \right| < |b|,$$

$$(iii) \quad 2(n+1)|a_2| + \sum_{m=2}^{\infty} \frac{(n+m-2)!}{n! (m-2)!} \left| (m-1)a_{m-1} - \frac{(n+m)(n+m-1)(m+1)}{m(m-1)} \right.$$

$$\left. a_{m+1} \right| < |b|, \text{ where } a_1 = 1,$$

$$(iv) \quad 2|(n+1)a_2 - a_1| + \sum_{m=2}^{\infty} \frac{(n+m-2)!}{n! (m-2)!} \left| (m-1)a_{m-1} - \frac{2(n+m-1)}{m-1} a_m \right. \\ \left. + \frac{(n+m)(n+m-1)(m+1)}{m(m-1)} a_{m+1} \right| < |b|, \text{ where } a_1 = 1.$$

PROOF. We prove the sufficiency of part (i) since the proofs of the remaining parts are similar to the proof of (i).

From (i) of Theorem 2.2, $f \in K_n(b)$ if f satisfies the condition

$$\operatorname{Re} \left\{ 1 + \frac{1}{b} [(D^n f(z))' - 1] \right\} > 0, \quad z \in E. \quad (2.11)$$

Condition (2.11) would be satisfied if

$$\left| \frac{1}{b} [(D^n f(z))' - 1] - 1 \right| < 2, \quad z \in E \quad (2.12)$$

is true. However upon substituting

$$(D^n f(z))' = 1 + \sum_{m=2}^{\infty} \frac{m(n+m-1)}{n! (m-1)!} a_m z^{m-1}$$

in (2.12) one needs only show

$$\left| \frac{1}{b} \sum_{m=2}^{\infty} \frac{m(n+m-1)!}{n! (m-1)!} a_m z^{m-1} - 1 \right| < 2, \quad z \in E. \quad (2.13)$$

Assuming (i) of this theorem we have

$$\left| \frac{1}{b} \sum_{m=2}^{\infty} \frac{m(n+m-1)!}{n! (m-1)!} a_m z^{m-1} - 1 \right| < \frac{1}{|b|} \sum_{m=2}^{\infty} \frac{m(n+m-1)!}{n! (m-1)!} |a_m| + 1 < 2.$$

Thus (2.13) is established and the proof of the sufficiency of part (i) is complete.

REMARK 2.3. For $n = 0$ and $b = 1$, Theorems 2.2 and 2.3 are reduced to theorems of Ozaki [6].

3. DISTORTION THEOREMS.

The objective of this section is to obtain some distortion theorems for the class $K_n(b)$. The radius of the largest disk $E(r) = \{z/|z| < r\}$, $0 < r < 1$ such that if $f \in K_n(b)$ then $f \in K_n(1)$ can be determined as a consequence of one of those results.

THEOREM 3.1. Let $f \in K_n(b)$, $n \in N_0$. Then for $|z| = r < 1$, and $|2b - 1| < 1$

$$\frac{1 - |2b - 1|r}{(1 + r)^3} < |(D^n f(z))'| < \frac{1 + |2b - 1|r}{(1 - r)^3}. \quad (3.1)$$

This result is sharp. An extremal function f is given by (2.1).

PROOF. Let $f \in K_n(b)$. Then (1.5) implies for some $g \in R_n$

$$\frac{z(D^n f(z))'}{D^n g(z)} = \frac{1 + (2b - 1)w(z)}{1 - w(z)}, \quad z \in E,$$

where $w \in A$ and $|w(z)| < |z|$ in E . This gives for $|z| = r < 1$

$$\frac{1 - |2b - 1|r}{1 + r} < \frac{|z(D^n f(z))'|}{D^n g(z)} < \frac{1 + |2b - 1|r}{1 - r}. \quad (3.2)$$

The definition of R_n implies $D^n g(z)$ is a starlike function. Hence by the well known bounds on functions which are starlike in E , we get for $|z| = r < 1$

$$\frac{r}{(1 + r)^2} < |D^n g(z)| < \frac{r}{(1 - r)^2}. \quad (3.3)$$

Using (3.2) together with (3.3) one can get (3.1) and the proof of the Theorem 3.1 is complete.

Taking (i) $n = 0$, and (ii) $n = 0$, $b = 1$ in Theorem 3.1, one can immediately obtain the following corollaries, respectively.

COROLLARY 3.1. If f is a close-to-convex function of complex order b where $|2b - 1| < 1$, then for $|z| = r < 1$

$$\frac{1 - |2b - 1|r}{(1 + r)^3} < |f'(z)| < \frac{1 + |2b - 1|r}{(1 - r)^3}.$$

COROLLARY 3.2. If f is a close-to-convex function then for $|z| = r < 1$,

$$\frac{1-r}{(1+r)^3} < |f'(z)| < \frac{1+r}{(1-r)^3}.$$

For the proof of Theorem 3.2, we need the following well known result [7; p. 84] concerning the class P of functions $p(z)$ which are regular in E such that $p(0) = 1$ and $\operatorname{Re} p(z) > 0$, $z \in E$.

LEMMA 3.1. Let $p \in P$. Then for $|z| = r < 1$,

$$\left| p(z) - \frac{1+r^2}{1-r^2} \right| < \frac{2r}{1-r^2}. \quad (3.4)$$

This result is sharp.

THEOREM 3.2. Let $f \in K_n(b)$, $n \in N_0$. Then for some $g \in R_n$ and for $|z| = r < 1$,

$$\left| \frac{z(D^n f(z))'}{D^n g(z)} - \frac{1 + (2b-1)r^2}{1-r^2} \right| < \frac{2|b|r}{1-r^2}. \quad (3.5)$$

This result is sharp. An extremal function is given in (2.1).

PROOF. $f \in K_n(b)$ implies that for some $g \in R_n$

$$1 + \frac{1}{b} \left[\frac{z(D^n f(z))'}{D^n g(z)} - 1 \right] = p(z), \quad z \in E,$$

where $p \in P$. Hence (3.5) can be obtained by substituting $p(z)$ in (3.4).

It is interesting to note that the result in Theorem 3.2 does not depend on the value of n . Also, it can be used to solve the problem concerning the radii of $K_n(b)$ in $K_n(1)$.

THEOREM 3.3. Let $n \in N_0$. If $f \in K_n(b)$, then $f \in K_n(1)$ for $|z| < r'$ where

$$r' = \frac{1}{|b| + \sqrt{|b|^2 - 2\operatorname{Re} b + 1}}.$$

This result is also sharp. An extremal function is given in (2.1).

PROOF. Let $f \in K_n(b)$. Then according to Theorem 3.2 there is some $g \in R_n$ such

that for $|z| = r < 1$, $\frac{z(D^n f(z))'}{D^n g(z)}$ lies in the closed disk with center

at $\frac{1 + (2b-1)r^2}{1-r^2}$ and radius $\frac{2|b|r}{1-r^2}$. It can be shown that this disk lies in the

right half plane if $r < r'$. This completes the proof of Theorem 3.3.

REMARK 3.1. Taking $n = 0$ in Theorem 3.3, one can see that, r' is the sharp radius of close-to-convexity for close-to-convex functions of complex order b .

REFERENCES

1. AOUP, M.K. and NASR, M.A., Starlike Functions of Complex Order b , J. Natural Sci. Math. 25(1), (1985), 1-12.
2. SINGH, S. and SINGH, S., Integrals of Certain Univalent Functions, Proc. Amer. Math. Soc. 77(3), (1979), 336-340.
3. AL-AMIRI, H.S., On Ruscheweyh Derivatives, Annales Polonic Math., 38(1), (1980), 88-94.
4. RUSCHEWEYH, S., Convolutions in Geometric Function Theory, Les Presses De l'Universite De Montreal (1982)
5. READE, M.O., On Close-to-Convex Univalent Functions, Michigan Math. J., 3, (1955), 59-62.
6. OZAKI, S., On The Theory of Multivalent Functions, Sci. Rep. Tokyo Burnika Paig. A2 (1935), 167-188.
7. GOODMAN, A.W., Univalent Functions 1, Marinar Publishing Company Inc.
8. CLUNIE, J., On Meromorphic Schlicht Functions, J. London Math. Soc. 34 (1952), 215-216.