A NOTE ON TAUBERIAN OPERATORS

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ABSTRACT. In this note we prove the existence of operators which are not Tauberian even though they satisfy properties about restrictions being Tauberian. The operators are defined on Banach spaces which contain a somewhat reflexive, non-reflexive subspace. This gives an answer to a question proposed by R. Neidinger [1].

KEY WORDS AND PHRASES. Tauberian Operators, and Semi-Fredholm Operators.

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1. INTRODUCTION.

Throughout this note E, F are infinite-dimensional Banach spaces over the real or complex field. All operators $T: E \longrightarrow F$ are assumed to be linear and continuous. Given $T \in L(E, F)$ the notation $T|_Z$ denotes the restriction of T to the subspace Z of E .

Recall that an operator $T \in L(E, F)$ is said to be semi-Fredholm if its null space $N(T)$, is finite-dimensional and its range space $R(T)$ is closed. Also, a Tauberian operator, as defined by D. Garling and A. Wilansky in [2], is a bounded linear operator $T \in L(E, F)$ such that T'' preserves the natural embedding of E into its double dual, i.e., $T''x \in F$ implies $x \in E$. Some relationships between these two classes of operators have been studied in [1], [3], [4] and [5]. In particular, if $R(T)$ is closed, then T is Tauberian if and only if $N(T)$ is reflexive.

It is well-known that the restriction of a semi-Fredholm operator to any closed subspace is again a semi-Fredholm operator. In the opposite direction it is

worthwhile to mention the following result that is basically due to T. Kato [6],

THEOREM 1 (c.f. [6]). Let E, F be infinite-dimensional Banach spaces. Assume that $T: E \rightarrow F$ is an operator such that every infinite-dimensional closed subspace Z of E contains an infinite-dimensional closed subspace W for which $T|_W$ is semi-Fredholm. Then T is semi-Fredholm.

It follows that in order to see that a given operator T is semi-Fredholm, it is enough to assure that its restriction to every closed subspace with a Schauder basis is semi-Fredholm.

Another related result is the following theorem due to R. Neidinger in which Banach spaces with no infinite-dimensional reflexive subspace are called "purely non-reflexive" spaces.

THEOREM 2 ([1], p. 26). Let E be a weakly sequentially complete Banach space and let $T \in L(E, F)$. Then T is Tauberian if (and only if) $T|_Z$ is semi-Fredholm for all purely non-reflexive closed subspaces Z of E .

In view of the preceding theorem, R. Neidinger raised the following question ([1], p. 139): If $T \in L(E, F)$, restricted to any purely non-reflexive closed subspace is semi-Fredholm, is T Tauberian? Indeed, the answer is positive if E is reflexive. Then, we assume that E is not reflexive. In this case there are some trivial situations for which the answer is negative (e.g., let E be a somewhat reflexive space, that is, every infinite-dimensional subspace of E contains an infinite-dimensional reflexive subspace, and let T be a finite rank operator). Our next example gives a negative answer to the question raised by R. Neidinger in a non-trivial situation.

EXAMPLE. Let J be the James space and let $T: J \times l^1 \rightarrow l^1$ be the operator defined by $T(x, y) = y$.

Since $R(T)$ is closed and $N(T) = J$ is not reflexive then, T is not Tauberian. Now, let Z be a purely non-reflexive closed subspace of $J \times l^1$. Since J is somewhat reflexive, $N(T|_Z) = N(T) \cap Z$ is finite-dimensional; otherwise, $N(T|_Z)$ would contain an infinite-dimensional reflexive subspace, which contradicts our assumption over Z .

Also, $N(T)$ and Z are totally incomparable Banach spaces (i.e., there exists no infinite-dimensional Banach space which is isomorphic to a subspace of $N(T)$ and to a subspace of Z). This implies that $N(T) + Z$ is closed in $J \times l^1$ [7] and hence, $T(Z) = R(T|_Z)$ is closed by the open mapping theorem.

Thus, $T|_Z$ is semi-Fredholm for all purely non-reflexive closed subspaces.

2. MAIN RESULTS.

Another related problem is as follows; we know that the restriction of a Tauberian operator to any closed subspace is again Tauberian. So, is Theorem 1 true for Tauberian operators instead of semi-Fredholm operators? The answer is obviously positive if, for instance E is reflexive or E is purely non-reflexive. However, we have,

THEOREM 3. Let E be an infinite-dimensional Banach space which contains an infinite-dimensional somewhat reflexive closed subspace M which is not reflexive.

Then there exists an infinite-dimensional Banach space F and a non-Tauberian surjective operator $T:E \longrightarrow F$ such that every infinite-dimensional closed subspace Z of E contains an infinite-dimensional closed subspace W for which $T|_W$ is Tauberian.

PROOF. First assume that E/M is infinite-dimensional and consider the quotient map $T:E \longrightarrow E/M$. It follows, as in the above example, that T is not Tauberian but that for every purely non-reflexive subspace Z of E then, $T|_Z$ is Tauberian. Now, assume that Z is not purely non-reflexive; in this case there exists an infinite-dimensional reflexive subspace $W \subset Z$. For this W , it is obvious that $T|_W$ is Tauberian.

If $\dim E/M < \infty$ then, E is itself somewhat reflexive and non-reflexive. Since E is not reflexive, there exists a bounded basic sequence (e_n) in E which is not weakly null [8]. Without loss of generality, (e_{2n}) is not weakly null, otherwise use (e_{2n-1}) . Let N be the closed linear span of (e_{2n}) . It follows that N is a non-reflexive closed subspace of E such that E/N is infinite-dimensional. Let us prove that the quotient map $T:E \longrightarrow E/N$ satisfies the conclusion. Given an infinite-dimensional closed subspace Z of E then, Z contains an infinite-dimensional reflexive subspace W ; it follows that $T|_W$ is Tauberian. But, on the other hand, T is not Tauberian because its null space N is not reflexive.

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