THE STRUCTURE OF HOMOMORPHISMS FROM BANACH ALGEBRAS OF DIFFERENTIABLE FUNCTIONS INTO FINITE DIMENSIONAL BANACH ALGEBRAS

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ABSTRACT: We show that the structure of continuous and discontinuous homomorphisms from the Banach algebra $C^n[0,1]$ of $n$ times continuously differentiable functions on the unit interval $[0,1]$ into finite dimensional Banach algebras is completely determined by higher point derivations.

KEY WORDS AND PHRASES. Banach algebras, homomorphisms, local algebras, singularity set, higher point derivations.

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0. Introduction.

It is well known that the Banach algebra $C^n[0,1]$ is generated by $z(t) = t$, $0 \leq t \leq 1$. Thus a continuous homomorphism $\nu$ of $C^n[0,1]$ into a Banach algebra $\mathfrak{A}$ is completely determined by $\nu(z)$. We are mainly interested in the structure of discontinuous homomorphisms $\nu$ from $C^n[0,1]$ into finite dimensional Banach algebras. In 1980 Bade, Curtis and Laursen [1] showed that these homomorphisms have a striking degree of continuity: the restriction of $\nu$ to $C^{2n}[0,1]$ is continuous with respect to the $C^{2n}$-norm. So, if we can obtain an explicit structure of continuous homomorphism $\nu$ from $C^n[0,1]$ into finite dimensional Banach algebras we may understand the behavior of discontinuous ones; that will be our approach to this problem.

1. Preliminaries.

Let $C^n[0,1]$ denote the algebra of all complex valued functions on $[0,1]$ which have $n$ continuous derivatives. It is well known that $C^n[0,1]$ is a Banach algebra under the norm

$$||f|| = \max_{t \in [0,1]} \sum_{k=0}^{n} \frac{|f^{(k)}(t)|}{k!}$$

whose structure space is $[0,1]$. We will need a characterization of the square of the closed primary ideals with finite codimension in $C^n[0,1]$. We use the notation

$$M_{n,k}(t_0) = \{f \in C^n[0,1] \mid f^{(j)}(t_0) = 0; j = 0,1,...,k\}.$$ 

These are precisely the closed ideals of finite codimension contained in the maximal ideal $M_{n,k}(t_0)$ of functions vanishing at $t_0$. Writing $M_{n,k}$ for $M_{n,k}(0)$ and setting $z(t) = t$, $0 \leq t \leq 1$, we have:
1.1 THEOREM. Let $n$ be a positive integer. Then

(i) $M^2_{n,0} = zM_{n,0} = f(f(0)) = f'(0) = 0$ and $f^{(n+1)}(0)$ exists,

(ii) $M^2_{n,k} = z^{k+1}M_{n,k}$, $1 \leq k \leq n-1$,

(iii) $M^2_{n,n} = z^nM_{n,n}$.

Part (i) is from [2, Example 3]. Part (ii) is due to Dales and McClure [3, Theorem 3.1]. The proof of part (iii) can be found in [4]. The squares of the closed primary ideals $M_{n,k}(t)$ at other points $t_0$ in $[0,1]$ are given exactly by similar formulas, where $z$ is replaced by $z - t_0$. We also need the following concepts and facts in automatic continuity theory.

1.2 DEFINITION. If $T: A \rightarrow \mathfrak{B}$ is a linear map and $A, \mathfrak{B}$ are Banach spaces, then the separating space of $T$, $\mathfrak{T}(T)$ is defined by

$$\mathfrak{T}(T) = \{y \in \mathfrak{B} | \exists \{x_n\} \subset A, x_n \rightarrow 0, \text{ and } T(x_n) \Rightarrow y\}.$$  

This space measures the discontinuity of $T$ because $\mathfrak{T}(T) = \{0\}$ if and only if $T$ is discontinuous, by the closed graph theorem. More detailed discussion on $\mathfrak{T}(T)$ can be found in [5].

1.3 DEFINITION. If $\mathfrak{A}, \mathfrak{B}$ are Banach algebras, and $T: \mathfrak{A} \rightarrow \mathfrak{B}$ is a homomorphism with separating space $\mathfrak{T}(T)$, then the continuity ideal of $T$, $\mathfrak{S}(T)$ is defined by

$$\mathfrak{S}(T) = \{z \in \mathfrak{A} | T(z)\mathfrak{T}(T) = \{0\}\}.$$  

Let $\mathfrak{B}$ be a Banach algebra and $\nu: C^n[0,1] \rightarrow \mathfrak{B}$ be a homomorphism. It is shown in [6,7] that the continuity ideal $\mathfrak{T}(\nu)$ has finite hull and contains the ideal $J(F)$ of all functions vanishing in neighborhoods of $F = \text{hull}(\mathfrak{T}(\nu))$. $F$ is called the singularity set of $\nu$.

1.4 THEOREM. Let $n$ be a positive integer and $\nu: C^n[0,1] \rightarrow \mathfrak{B}$ be a discontinuous homomorphism with singularity set $F = \{0\}$. Consider the following statements:

(a) $\mathfrak{T}(\nu)$ is finite dimensional,
(b) $\mathfrak{T}(\nu)$ has finite codimension,
(c) $\mathfrak{T}(\nu)$ is closed and contains $M_{n,n-1}$,
(d) $\mathfrak{T}(\nu)^2 = \{0\}$,
(e) $z^n \in \mathfrak{T}(\nu)$,
(f) $\nu$ is continuous on $M^2_{n,n} = z^nM_{n,n}$ for the graph norm $\|f\| = \|f\| + \|\nu f\|$,
(g) $\nu$ is $C^{2n}$-continuous (i.e. the restriction of $\nu$ to $C^{2n}$ is continuous with respect to the $C^{2n}$-norm).

We have the following implications:

(a) $\Rightarrow$ (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (e) $\Leftrightarrow$ (f) $\Rightarrow$ (g).

For the proof see [1].

2. Algebraic results.

Let $\nu$ be a homomorphism of $C^n[0,1]$ into a finite dimensional Banach algebra $\mathfrak{B}$. We may assume that $\nu$ is onto by considering $\nu: C^n[0,1] \rightarrow \nu(C^n[0,1])$. We shall reduce the study of $\nu$ to the case where the range space is a finite dimensional local algebra.
Let \( \mathfrak{a} \) be the range of \( \nu \). Since \( \mathfrak{a} \) is a finite dimensional commutative algebra with a unit \( e \), the Wedderburn principal theorem states that \( \mathfrak{a} = \mathcal{A} + \mathfrak{a} \), where \( \mathfrak{a} \) is the radical of \( \mathfrak{a} \). Now \( \mathcal{A} \) is a semisimple commutative algebra with unit. By the Wedderburn structure theorem for finite dimensional algebras we can write

\[
\mathcal{A} = e_1 \mathcal{A} \oplus \ldots \oplus e_m \mathcal{A},
\]

where

\[
e_i e_j = 0, \quad i \neq j,
\]

\[
e_i^2 = e_i, \quad i = 1, 2, \ldots, m,
\]

\[
e = e_1 + \ldots + e_m,
\]

and \( e_i \mathcal{A} \) is a simple commutative algebra with unit \( e_i \), so that \( e_i \mathcal{A} \cong \mathbb{C} \). Thus we can write

\[
\mathfrak{a} = e_1 \mathfrak{a} \oplus \ldots \oplus e_m \mathfrak{a},
\]

where each \( e_i \mathfrak{a} \) is a local algebra which may be isomorphic to \( \mathbb{C} \). Moreover \( \nu = e_1 \nu + \ldots + e_m \nu \) and each \( e_i \nu \) is a homomorphism of \( C^n[0,1] \) onto \( e_i \mathfrak{a} \). If \( e_i \mathfrak{a} \cong \mathbb{C} \) then \( e_i \nu \) is just a multiplicative functional and is of the form \( e_i \nu(f) = f(t_0) e_i \) for some \( t_0 \) in \([0,1]\). It remains to consider the case when \( e_i \mathfrak{a} \) is a local algebra which is not isomorphic to \( \mathbb{C} \). Our next objective is to characterize the kernel of \( \nu \).

2.1 LEMMA. Let \( \nu \) be a homomorphism of \( C^n[0,1] \) onto a finite dimensional algebra. If \( t_0 \in \hull(\ker \nu) \) then

(i) \( (z - t_0)^m \in \ker \nu \) for some positive integer \( m \),

(ii) The ideal \( J(t_0) \) of all functions vanishing in neighborhoods of \( t_0 \) is contained in \( \ker \nu \).

PROOF: (i) Suppose that \( \nu(z - t_0) \) is not nilpotent. Then \( \nu(z - t_0) \) is invertible so that there exists \( g \) in \( C^n[0,1] \) such that \( \nu(z - t_0) \nu(g) = \nu(1) \). Thus \( (z - t_0)g = 1 + f \) for some \( f \) in \( \ker \nu \). But \( (z - t_0)g \in \mathbb{M}_{n,0}(t_0) \) and \( f \in \ker \nu \subset \mathbb{M}_{n,0}(t_0) \) so that \( f \in \mathbb{M}_{n,0}(t_0) \). This is a contradiction.

(ii) Let \( f \in J(t_0) \). Choose \( g \in J(t_0) \) such that \( g \) is identically one on the support of \( f \). We claim that \( \nu(g) \) is nilpotent. Suppose not, then \( \nu(g) \) is invertible and \( e = \nu(h) \) for some \( h \in J(t_0) \). So \( f = f + v \) for some \( v \in \ker \nu \). This is a contradiction since \( h \in J(t_0) \subset \mathbb{M}_{n,0}(t_0) \) and \( v \in \mathbb{M}_{n,0}(t_0) \). Thus \( g^m \in \ker \nu \) for some \( m \) and we have \( f = fg^m \in \ker \nu \).

Immediately following this lemma we have:

2.2 COROLLARY. Let \( \nu \) be a homomorphism of \( C^n[0,1] \) onto a finite dimensional local algebra. The hull of \( \ker \nu \) consists of exactly one point \( t_0 \) and therefore the singularity set \( F = \hull(\ker \nu) \subset \{ t_0 \} \).

PROOF: Let \( t_0 \) and \( t_1 \) be in \( \ker(\ker \nu) \). By 2.1 there exist positive integers \( m_0 \) and \( m_1 \) such that \( \nu(z - t_0)^{m_0} = 0 \) and \( \nu(z - t_1)^{m_1} = 0 \). Then \( \nu[(z - t_1) - (z - t_0)]^{m_0 + m_1} = 0 \) so that \( t_1 - t_0 = 0 \). Thus \( \hull(\ker \nu) = \{ t_0 \} \) for some \( t_0 \) in \([0,1]\). Since \( \ker \nu \subset \mathfrak{a}(\nu) \) it follows that \( \hull(\ker \nu) \subset \{ t_0 \} \).

Without loss of generality we shall take \( \hull(\ker \nu) \) to be \( \{ 0 \} \). With this assumption we now describe \( \nu \) for the case when it is continuous.
2.3 THEOREM. Let \( \nu \) be a continuous homomorphism of \( C^n[0,1] \) onto a finite dimensional local algebra with \( \text{hull}(\ker \nu) = \{0\} \). There exists a positive integer \( k \leq n + 1 \) such that
\[
\nu(f) = \sum_{i=1}^{k-1} \delta_i(f) \nu(x)^i,
\]
where \( \delta_i(f) = \frac{f^{(i)}(0)}{i!} \).

PROOF: Since \( \nu \) is continuous, \( \ker \nu \) is a closed primary ideal of finite codimension. Thus \( \ker \nu = M_{n,k-1} \) for some \( k \leq n + 1 \). Let \( f \in C^n[0,1] \), we write \( f = \sum_{i=0}^{k-1} \delta_i(f) x^i + Rf \), where \( Rf \in M_{n,k-1} = \ker \nu \). Then
\[
\nu(f) = \sum_{i=0}^{k-1} \delta_i(f) \nu(x)^i.
\]
The sequence of linear functionals \( \delta_0, \delta_1, \ldots, \delta_{k-1} \) is called a continuous higher point derivation of order \( k - 1 \) on \( C^n[0,1] \) at \( a_0 \). We refer to [3] for a complete description of the order and continuity properties of higher point derivations on \( C^n[0,1] \).

3. The structure of discontinuous homomorphisms of \( C^n[0,1] \) onto finite dimensional local algebras.

We now turn our attention to discontinuous homomorphisms \( \nu \) of \( C^n[0,1] \) onto a finite dimensional local algebra \( \mathfrak{B} \) with \( \text{hull}(\ker \nu) = \{0\} \). First we characterize \( \ker \nu \).

3.1 LEMMA. \( \overline{\ker \nu} = \nu^{-1}(\mathfrak{J}(\nu)) \).

PROOF: Let \( f \in \ker \nu \), there exists \( \{f_m\} \subset \ker \nu \) with \( f_m \to f \). Then \( f - f_m \to 0 \) and \( \nu(f - f_m) \to \nu(f) \) so that \( \nu(f) \in \mathfrak{J}(\nu) \). Hence \( \overline{\ker \nu} \subseteq \nu^{-1}(\mathfrak{J}(\nu)) \).

Now let \( f \in \nu^{-1}(\mathfrak{J}(\nu)) \). By definition of \( \mathfrak{J}(\nu) \), there exists \( \{f_k\} \subset C^n[0,1] \) with \( f_k \to 0 \) and \( \nu(f_k) \to \nu(f) \). Since \( \overline{\nu} \) has finite codimension in \( C^n[0,1] \), there exists a subspace \( V \) with \( \dim V < \infty \) such that \( C^n[0,1] = \overline{\ker \nu} \oplus V \). So we can write \( f_k = g_k + v_k \) where \( g_k \in \ker \nu \) and \( v_k \in V \). But \( f_k \to 0 \) so that \( g_k \to 0 \) and \( v_k \to 0 \). Since \( \dim V < \infty \), \( \nu(v_k) \to 0 \) so that \( \nu(g_k) = \nu(f_k) - \nu(v_k) \to \nu(f) \). Again we can write \( \ker \nu = \ker \nu \oplus W \), where \( \dim W < \infty \), so \( g_k = h_k + w_k \) where \( h_k \in \ker \nu \) and \( w_k \in W \). Then \( \nu(g_k) = \nu(w_k) \to \nu(f) \) so that \( \nu(f) \in \mathfrak{J}(W) \). Thus \( f \in W \oplus \ker \nu = \ker \nu \) and we conclude that \( \nu^{-1}(\mathfrak{J}(\nu)) \subseteq \ker \nu \).

3.2 LEMMA. Let \( k \) be the integer for which \( \overline{\ker \nu} = M_{n,k-1} \), \( k \leq n + 1 \). Then \( M^2_{n,k-1} \subset \ker \nu \) and
\[
\mathfrak{B} = \text{span}\{e, \nu(z), \ldots, \nu(x)_{k-1}\} \oplus \mathfrak{J}(\nu).
\]

PROOF: The first statement is clear since \( \overline{\ker \nu} \) is a closed primary ideal of finite codimension. By 3.1 \( \nu(M_{n,k-1}) = \mathfrak{J}(\nu) \). Let \( f, g \in M_{n,k-1} \), then \( \nu(fg) = \nu(f) \nu(g) \in \mathfrak{J}(\nu)^2 = \{0\} \) by 1.4. Since \( C^n[0,1] = \text{span}\{1, z, \ldots, z^{k-1}\} \oplus \overline{\ker \nu} \) we have
\[
\mathfrak{B} = \nu(C^n[0,1]) = \text{span}\{e, \nu(z), \ldots, \nu(x)_{k-1}\} \oplus \mathfrak{J}(\nu)
\]
by 3.1. To see that the sum is direct let
\[
b = a_0 e + a_1 \nu(z) + \ldots + a_{k-1} \nu(x)_{k-1} \in \mathfrak{J}(\nu)
\]
and suppose \( b \neq 0 \). Let \( j \) be the smallest integer such that \( a_j \neq 0 \), then
\[
\nu(z)^j \left( \sum_{i=j}^{k-1} a_i \nu(z)^{i-j} \right) \in \mathcal{I}(\nu).
\]

But \( \sum_{i=j}^{k-1} a_i \nu(z)^{i-j} \) is invertible since \( a_j \neq 0 \), so \( \nu(z)^j \in \mathcal{I}(\nu) \). By \( 3.1 \), \( x^j \in \ker \nu = M_{n,k-1} \). This is a contradiction since \( j \leq k - 1 \).

We are now in position to describe discontinuous homomorphisms.

3.3 THEOREM. Let \( \nu \) be a discontinuous homomorphism of \( C^n[0,1] \) onto a finite dimensional local algebra \( \mathfrak{A} \) with \( \text{hull}(\ker \nu) = \{0\} \) and \( \ker \nu = M_{n,k-1} \). There exist \( b_1, \ldots, b_m \) in \( \mathcal{I}(\nu) \) and discontinuous linear functionals \( \gamma_1, \ldots, \gamma_m \) on \( C^n[0,1] \) which vanish on polynomials and on the principal ideal \( z^k C^n[0,1] \) such that

\[
\nu(f) = \sum_{l=0}^{k-1} d_l(f) \nu(z)^l + \sum_{l=0}^{i_1} \gamma_l(z^{k_1-l} f) \nu(z)^l b_1 + \cdots + \sum_{l=0}^{i_m} \gamma_m(z^{i_m-l} f) \nu(z)^l b_m,
\]

\( 0 \leq k_1, i_1, \ldots, i_m \leq k - 1 \),

where \( d_1, \ldots, d_{k+k_1} \) is a higher point derivation at 0 and the linear functionals \( \gamma_j \) defined by \( \gamma_j(f) = \gamma_j(z^j f) \), \( j = 1, \ldots, m \), are discontinuous point derivations at 0.

PROOF: Since \( z^k \in M_{n,k-1} = \ker \nu = \nu^{-1}(\mathcal{I}(\nu)) \) and \( \mathcal{I}(\nu)^2 = \{0\} \), the multiplication operator \( \nu(z): \mathcal{I}(\nu) \to \mathcal{I}(\nu) \) is nilpotent of index less than or equal to \( k \). So we may choose a basis \( B \) for \( \mathcal{I}(\nu) \) of the form

\[
B = \{ \nu(z)^{k_1}, \nu(z)^{k+k_1}, b_1, \nu(z)b_1, \ldots, \nu(z)^i b_1, \ldots, b_m, \nu(z)b_m, \ldots, \nu(z)^i b_m \}
\]

where \( 0 \leq k_1, i_1, \ldots, i_m \leq k - 1 \). Let \( f \in C^n[0,1] \). Consider the Taylor expansion

\[
f = \sum_{i=0}^{k-1} \delta_i(f) x^i + Rf,
\]

where \( Rf \in M_{n,k-1} = \ker \nu \). Since \( \nu(Rf) \in \mathcal{I}(\nu) \) we can write

\[
\nu(f) = \sum_{l=0}^{k-1} \delta_l(f) \nu(z)^l + \sum_{l=k}^{k+k_1} \delta_l(f) \nu(z)^l + \sum_{l=0}^{i_1} \gamma_{l}(\nu(z)^l b_1 + \cdots + \sum_{l=0}^{i_m} \gamma_m(z^{i_m} l) \nu(z)^l b_m
\]

We make the following observations:

(i) The coefficient functionals \( \delta_1, \ldots, \delta_{k+k_1}, \gamma_{1,1}, \ldots, \gamma_{1,i_1+1}, \ldots, \gamma_{m,1}, \ldots, \gamma_{m,i_m+1} \) are discontinuous. To see this, consider \( \gamma_{1,1} \). Since \( b_1 \in \mathcal{I}(\nu) \), there exist \( f_1 \in 0 \) in \( C^n[0,1] \) with \( \nu(f_1) = b_1 \). We have

\[

\nu(f_1)^{-1} b_1 = \sum_{l=0}^{k-1} \delta_l(f_1) \nu(z)^l + \sum_{l=k}^{k+k_1} \delta_l(f_1) \nu(z)^l + (\gamma_{1,1}(f_1) - 1) b_1 +

\]

\[

+ \sum_{l=1}^{i_1} \gamma_{1,i_1}(f_1) \nu(z)^l b_1 + \sum_{l=0}^{i_2} \gamma_{2,i_2+1}(f_1) \nu(z)^l b_2 + \cdots + \sum_{l=0}^{i_m} \gamma_{m,i_m+1}(f_1) \nu(z)^l b_m
\]
Since \( \nu(f_j) - b_i \Rightarrow 0 \) we must have all coefficients tending to zero as \( j \to \infty \), in particular
\[
\lim_{j \to \infty} \gamma_{1,1}(f_j) = 1,
\]
which implies that \( \gamma_{1,1} \) is discontinuous. The same argument works for the other coefficient functionals \( d_i, \ldots, d_{k+1}, \gamma_{1,2}, \ldots, \gamma_{1,1+k}, \ldots, \gamma_{m,1}, \ldots, \gamma_{m,1+m+1} \).

(ii) Since \( z^kM_{n_k} - 1 = M_{n_k}^2 \) all the above functionals vanish on \( z^kM_{n_k} - 1 \).

For a notational purpose we set \( d_i = \delta_i \) for \( i = 0, 1, \ldots, k - 1 \). Let \( f, g \in C^k[0,1] \). Using the fact that \( \nu(z^{k+k+1}) = 0 \) and \( \delta(\nu)^2 = 0 \) (by 1.4), we have
\[
\nu(f) \nu(g) = \sum_{l=0}^{k} \sum_{j=0}^{k} d_j(f) d_j(g) \nu(z^l) + \sum_{l=0}^{k} \sum_{j=0}^{k} d_j(f) \gamma_{1,1+l} - j(g) + \gamma_{1,1+l} - j(f) d_j(g) \nu(z^l) b_1 + \ldots
\]
\[
+ \sum_{l=0}^{k} \sum_{j=0}^{k} d_j(f) \gamma_{m,1+l} - j(g) + \gamma_{m,1+l} - j(f) d_j(g) \nu(z^l) b_m
\]
Since \( \nu(f) \nu(g) = \nu(fg) \) we have
\[
(iii) \quad d_i(fg) = \frac{1}{j} d_j(f) d_j(g) \quad \text{for } 0 \leq l \leq k+k+1, \text{ so } d_i, \ldots, d_{k+1} \text{ is a higher point derivation at } 0.
\]

(iv) For \( j = 1, \ldots, m \) and \( l = 1, \ldots, i \), we have \( \gamma_{j,1+l}(x^i) = 0 \), \( i = 0, 1, 2, \ldots \). Because \( \gamma_{j,1+l}(x^i) = 0 \), for \( i = 0, 1, \ldots, k+k+1 \), since \( d_j(x^i) = 0 \) if \( i \neq j \), \( d_j(x^i) = 1 \) if \( i = j \), and \( \text{span}\{e, \nu(z), \ldots, \nu(z)^{k+1} \} \oplus \text{span} B = \mathbb{R}. \gamma_{j,1+l}(x^i) = 0 \), for \( i \geq k+k+1 \) since \( \nu(z)^{k+k+1} = 0 \).

Combining (ii) and (iv) we see that all the \( \gamma_{j,1+l} \) vanish on \( z^kC^k[0,1] \).

(v) For \( j = 1, \ldots, m \) and \( l = 1, \ldots, i \), we have
\[
\gamma_{j,1+l}(fg) = \sum_{s=0}^{l} \delta_i(f) \gamma_{j,1+s} - j(g) + \gamma_{j,1+s} - j(f) \delta_s(g)
\]
so that
\[
\gamma_{j,1+l}(-i - f) = \gamma_{j,1+l}(f), \quad f \in C^k[0,1], j = 1, \ldots, m.
\]
We take \( \gamma_j = \gamma_{j,1+j}, \quad j = 1, \ldots, m \). Letting \( i = 0 \) in (v), we note that the linear functionals \( \theta_j, \quad (j = 1, \ldots, m) \), defined by \( \theta_j(f) = \gamma_j, \quad (j = 1, \ldots, m) \), are discontinuous point derivations at 0.

REFERENCES


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