BOUNDED ANALYTIC FUNCTIONS AND THE LITTLE BLOCH SPACE

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ABSTRACT. The radial limits of the weighted derivative of an bounded analytic function is considered.

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1. INTRODUCTION.

Let D denote the unit disc of the complex plane C and let H^∞ denote the space of bounded analytic functions on D . An analytic function on D is called a Bloch function if $\sup_{z \in D} |f'(z)| (1-|z|^2) < \infty$. The space $\mathfrak{B}$ of Bloch functions is a Banach space with norm

$$\|f\|_{\mathfrak{B}} = |f(0)| + \sup_{z \in D} |f'(z)| (1-|z|^2).$$

A Bloch function is in the little Bloch space $\mathfrak{B}_0$, if $f'(z) (1-|z|^2) \rightarrow 0$ as $|z| \rightarrow 1^-$. An immediate consequence of Schwartz lemma (see for example [2], Lemma 1.2) is that $H^\infty \subset \mathfrak{B}$, however it is well known (see section 3 for an explicit example) that $H^\infty \not\subset \mathfrak{B}_0$. The main result of this paper is to show that, if $f \in H^\infty$ then

$$f'(r e^{i\theta}) (1-r^2) \rightarrow 0 \text{ for almost all } \theta \text{ as } r \rightarrow 1^-.$$

2. APPROXIMATE IDENTITY.

In this section we establish an approximate identity akin to the Poisson Kernel.

LEMMA 1. Let $0 < r < 1$, $t \in \mathbb{R}$ and

$$\varphi(r, t) = \frac{(1-r^2)^3}{(1+r^2)} \frac{1}{(1 - 2r \cos t + r^2)^2}$$

Then $\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, t) dt = 1.$

PROOF: Let $P_r(t) = \frac{1-r^2}{1 - 2r \cos t + r^2} = 1 + 2 \sum_{n=1}^{\infty} r^n \cos nt$ be the Poisson kernel. As usual let L^2 be the Lebesgue 2-space on $[0, 2\pi]$. Then,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, t) dt &= \frac{(1-r^2)}{(1+r^2)} \|P_r(t)\|_{L^2}^2 \\ &= \frac{(1-r^2)}{(1+r^2)} \left\langle 1 + 2 \sum_{n=1}^{\infty} r^n \cos nt, 1 + 2 \sum_{n=1}^{\infty} r^n \cos nt \right\rangle \\ &= \frac{1-r^2}{1+r^2} \left(1 + 2 \sum_{n=1}^{\infty} r^{2n} \right) = 1. \end{aligned}$$

LEMMA 2. Let μ be a complex measure on $[-\pi, \pi]$ and suppose the derivative $D\mu(\theta)$ exists for some point θ . Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, \theta-t) d\mu(t) \rightarrow D\mu(\theta) \text{ as } r \rightarrow 1-.$$

PROOF: The usual approximate identity proof with the Poisson kernel $P_r(\theta-t)$ (see for example [1] page 4) works for $\varphi(r, \theta-t)$ as well.

Without loss of generality we may assume that $\theta = 0$. Let $A = D\mu(0)$ and

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, \theta-t) d\mu(t).$$

then

$$\begin{aligned} u(re^{i\theta}) - A &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, t) [d\mu(t) - A dt] \\ &= \frac{1}{2\pi} [\varphi(r, t) [\mu(t) - At]]_{-\pi}^{\pi} \\ &\quad - \frac{1}{2\pi} \int_{-\pi}^{\pi} (\mu(t) - At) \frac{\partial \varphi}{\partial t} dt \end{aligned}$$

where $\mu(t)$ is the function of Bounded Variation associated with μ . Note that the first term tends to 0 as $r \rightarrow 1^-$. Fix $\delta > 0$ and let $\delta \leq |t| \leq \pi$.

Then

$$\left| -\frac{\partial \varphi}{\partial t}(r, t) \right| \leq \frac{(1-r^2)^3}{16r^2} \frac{1}{\sin^6 \delta/2}.$$

Hence for each $\delta > 0$,

$$u(re^{i\theta}) - A - I_\delta \rightarrow 0$$

as $r \rightarrow 1^-$, where

$$I_\delta = -\frac{1}{2\pi} \int_{-\delta}^{\delta} (\mu(t) - At) \frac{\partial \varphi}{\partial t} dt = \frac{1}{\pi} \int_0^\delta \left[\frac{\mu(t) - \mu(-t)}{2t} - A \right] t \left(-\frac{\partial \varphi}{\partial t} \right) dt$$

Given $\epsilon > 0$, choose $\delta > 0$ such that

$$\left| \frac{\mu(t) - \mu(-t)}{2t} - A \right| < \frac{\epsilon}{2} \quad \text{for } 0 < t \leq \delta.$$

$$\text{Then } |I_\delta| \leq \frac{\epsilon}{2\pi} \int_0^\pi t \left| -\frac{\partial \varphi}{\partial t} \right| dt.$$

$$\text{But then } \frac{\partial}{\partial t} \varphi(r, t) = -\frac{(1-r^2)^3}{(1+r^2)} \frac{4r \sin t}{(1 - 2r \cos t + r^2)^3}.$$

$$\begin{aligned} \text{Hence } |I_\delta| &\leq \frac{\epsilon}{2\pi} \int_0^\pi t \left(-\frac{\partial \varphi}{\partial t}(r, t) \right) dt \\ &= -\frac{\epsilon}{2\pi} [t \varphi(r, t)]_0^\pi + \frac{\epsilon}{2\pi} \int_0^\pi \varphi(r, t) dt \\ &\leq \frac{\epsilon}{2\pi} \int_{-\pi}^\pi \varphi(r, t) dt = \epsilon. \end{aligned}$$

Now we are ready to prove the main result of this paper. We may recall that if $f \in H^\infty$ then the radial limits, $\lim_{r \rightarrow 1^-} f(re^{i\theta}) = f(e^{i\theta})$ exists for almost all θ and

that

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^\pi \frac{(1-r^2)}{1 - 2r \cos(\theta-t) + r^2} f(e^{it}) dt.$$

Taking derivatives with respect to r , we get

$$e^{i\theta} f'(re^{i\theta}) = I_1(re^{i\theta}) - I_2(re^{i\theta}),$$

where $I_1(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{-2r}{1 - 2r \cos(\theta-t) + r^2} f(e^{it}) dt$ and

$I_2(re^{i\theta}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{(1-r^2)(r - \cos(\theta-t))}{(1 - 2r \cos(\theta-t) + r^2)^2} f(e^{it}) dt$. Note that $(1-r) I_1(re^{i\theta}) \rightarrow -f(e^{i\theta})$ for almost all θ as $r \rightarrow 1^-$. Also

$$\begin{aligned} I_2(1-r) &= \frac{1}{\pi} (1-r) \int_{-\pi}^{\pi} \frac{(1-r^2)(-\cos(\theta-t) + 1)}{(1 - 2r \cos(\theta-t) + r^2)^2} f(e^{it}) dt \\ &\quad - \frac{(1-r)^2}{\pi} \int_{-\pi}^{\pi} \frac{(1-r^2)}{(1 - 2r \cos(\theta-t) + r^2)^2} f(e^{it}) dt. \end{aligned}$$

The first term in $I_2(1-r)$ is dominated by $(1-r) \|f\|_{\infty}$ and hence tends to zero as $r \rightarrow 1^-$. However the second term of $I_2(1-r)$ is

$$= - \frac{2(1+r^2)}{(1+r)^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(r, \theta-t) f(e^{it}) dt \rightarrow -f(e^{i\theta})$$

for almost all θ as $r \rightarrow 1^-$ by Lemma 2.

The Proof is complete.

3. A BLASCHKE PRODUCT NOT IN $\mathfrak{B}_0$.

In this section, for completeness sake we give an explicit example of an bounded analytic function which is not in $\mathfrak{B}_0$.

First we state an elementary lemma, whose proof we omit.

LEMMA 3. Let $0 < r < 1$. Then

- (a) If $0 < x < r < 1$, then $-\ln(1-x) < \frac{x}{1-r}$
- (b) $\frac{1-x}{1-xr}$ decreases on $-\infty < x < \frac{1}{r}$
- (c) $\frac{1+x}{1-xr}$ increases on $-\infty < x < \frac{1}{r}$.

LEMMA 4. Let $\alpha_n = 1 - \frac{1}{2^n}$ and $2\beta_n = \alpha_n + \alpha_{n+1}$ with $n \geq 1$.

Then $\lim_{m \rightarrow \infty} \prod_{n=1}^{\infty} \left| \frac{\beta_m - \alpha_n}{1 - \overline{\alpha_n} \beta_m} \right| \geq c > 0$ for some c .

PROOF: Fix $m \geq 1$. We first show that

$$\sum_{m+1}^{\infty} -\ln \left| \left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha_n} \beta_m} \right] \right| \leq 32.$$

For,

$$\begin{aligned} \sum_{m+1}^{\infty} -\ln \left| \left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha_n} \beta_m} \right] \right| &= \sum_{m+1}^{\infty} -\ln \left(1 - \frac{(1-\alpha_n)(1+\beta_m)}{1 - \overline{\alpha_n} \beta_m} \right) \\ &\leq \left[\frac{1 - \alpha_{m+1} \beta_m}{\alpha_{m+1} - \beta_m} \right] \sum_{m+1}^{\infty} \frac{(1-\alpha_n)(1+\beta_m)}{1 - \beta_m}. \quad (\text{use Lemma 3 (a) and (b)}). \end{aligned}$$

Then

$$\sum_{m+1}^{\infty} -\ln \left| \left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha_n} \beta_m} \right] \right| \leq \frac{(1-|\alpha_m|^2)}{\frac{1}{2}(\alpha_{m+1} - \alpha_m)} \frac{2}{(1-\beta_m)} \sum_{m+1}^{\infty} \frac{1}{2^n} \leq 32.$$

Now applying Lemma 3(c) to $\frac{1+x}{1-x\beta_m}$, we have $\frac{(1-\beta_m)(1+\alpha_n)}{1 - \overline{\alpha_n} \beta_m} \leq \frac{(1-\beta_m)(1+\alpha_m)}{1 - \overline{\alpha_m} \beta_m}$

for $1 \leq n \leq m$.

Thus,

$$\begin{aligned} \sum_{n=1}^m -\ln \left| \left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha_n} \beta_m} \right] \right| &= \sum_{n=1}^m -\ln \left[1 - \frac{(1-\beta_m)(1+\alpha_n)}{1 - \overline{\alpha_n} \beta_m} \right] \\ &\leq \left[\frac{1 - \alpha_m \beta_m}{\beta_m - \alpha_m} \right] \sum_{n=1}^m \frac{(1-\beta_m)(1+\alpha_n)}{1 - \overline{\alpha_n} \beta_m} \text{ by Lemma 3 (a), (c)}. \end{aligned}$$

Hence;

$$\begin{aligned} \sum_{n=1}^m -\ln \left| \left[\frac{\alpha_n - \beta_m}{1 - \overline{\alpha_n} \beta_m} \right] \right| &\leq \left[\frac{1 - \alpha_m \beta_m}{\beta_m - \alpha_m} \right] (1-\beta_m) \sum_{n=1}^m \frac{2}{1-\alpha_n} \\ &\leq \frac{1 - \alpha_m^2}{\frac{1}{2}(\alpha_{m+1} - \alpha_m)} (1-\beta_m) \sum_{n=1}^m 2^{n+1} \leq 64. \end{aligned}$$

Now Lemma 3 follows.

COROLLARY 1. Let b be the Blaschke product with zeros $\{a_n : n \geq 1\}$. Then

$$b'(r)(1-r) \rightarrow 0 \text{ as } r \rightarrow 1-.$$

PROOF: Let $c > \overline{\lim}_{r \rightarrow 1-} |b'(r)| (1-r)$.

Then for sufficiently large n ,

$$|b'(a_n)| \leq c \int_{a_n}^{\beta_n} \frac{1}{1-r} dr \leq c \int_{a_n}^{a_{n+1}} \frac{1}{1-r} dr = c \ln 2.$$

Hence by Lemma 4,

$$\overline{\lim}_{r \rightarrow 1-} |b'(r)| (1-r) \neq 0.$$

REFERENCES

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2. GARNETT, John B., Bounded Analytic Functions, Academic Press, 1981.