A DUALITY THEOREM FOR SOLUTIONS OF ELLIPTIC EQUATIONS

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ABSTRACT. Let L be a second order linear partial differential operator of elliptic type on a domain Ω of $\mathbb{R}^m$ with coefficients in $C^\infty(\Omega)$. We consider the linear space of all solutions of 'the equation $Lu = 0$ on Ω with the topology of uniform convergence on compact subsets and describe the topological dual of this space. It turns out that this dual may be identified with the space of solutions of an adjoint equation "near the boundary" modulo the solutions of this adjoint equation on the entire domain.

1. INTRODUCTION AND PRELIMINARY NOTIONS.

Let $H(R)$ denote the linear space of harmonic functions on a noncompact Riemann surface R with the topology of uniform convergence on compact subsets and let us consider the linear space $H(R)'$ of all continuous linear functionals on $H(R)$. Nakai and Sario [1] showed that this dual can be identified with a quotient space of harmonic functions as follows:

$$H(R)' = H(\infty_R)/H(R),$$

where ∞_R represents the Alexandroff ideal boundary point of R and where $H(\infty_R)$ is the space of germs of functions "harmonic at ∞_R".

In this work we replace $H(R)$ by the space of solutions of a linear partial differential equation and look for a similar description of the corresponding topological dual. We now introduce these solutions.

Let Ω be a domain (open connected subset) of $\mathbb{R}^m$ with $m \geq 2$. We consider the following linear partial differential operator with real valued coefficients $a_{ij} = a_{ji}$, β_i and c in $C^\infty(\Omega)$,

$$L = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} \beta_i(x) \frac{\partial}{\partial x_i} + c(x) . \qquad (1.1)$$

We assume that L is of *elliptic* type in Ω. This means that

$$\sum_{i,j=1}^{m} a_{ij}(x) \xi_i \xi_j > 0 , \qquad (1.2)$$

for each $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m \backslash \{0\}$ and for each $x \in \Omega$. From (1.2) we see that for every compact subset K of Ω, there exists a constant $\lambda = \lambda(K) > 0$ such that

$$\lambda \sum_{i=1}^{m} \xi_i^2 \le \sum_{i,j=1}^{m} a_{ij}(x) \xi_i \xi_j \le \lambda^{-1} \sum_{i=1}^{m} \xi_i^2 , \qquad (1.3)$$

for each $\xi \in \mathbb{R}^m$ and for each $x \in K$.

WEYL'S LEMMA. [2, page 271]. Let $f \in C^\infty(\Omega)$. If $u \in C^0(\Omega)$ satisfies,

$$\int_\Omega u(x) L \varphi(x) dx = \int_\Omega f(x) \varphi(x) dx , \qquad (1.4)$$

for every $\varphi \in C_c^\infty(\Omega)$, then $u \in C^\infty(\Omega)$.

Therefore if $u \in C^0(\Omega)$ satisfies (1.4) for every $\varphi \in C_c^\infty(\Omega)$ then, at each point $x \in \Omega$, u satisfies the *adjoint equation* $L^* u(x) = f(x)$ where,

$$L^* u(x) = \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j} (a_{ij}(x) u(x)) -$$

$$\qquad (1.5)$$

$$\sum_{i=1}^{m} \frac{\partial}{\partial x_i} (\beta_i(x) u(x)) + c(x) u(x) .$$

The operator L^* defined by (1.5) is called the *adjoint* of L.

We denote by $L(\Omega)$ the set of all solutions of $Lu = 0$ on Ω and by $L^*(\Omega)$ the set of all solutions of $L^* u = 0$ on Ω. The set $L(\Omega)$ is a locally convex linear subspace of the locally convex topological linear space $C^0(\Omega)$. We endow $L(\Omega)$ with the topology of uniform convergence on compact subsets of Ω and we denote by $L(\Omega)'$ the linear space of all continuous linear functionals on $L(\Omega)$.

We wish to describe $L(\Omega)'$. In the next sections we shall identify this dual space with a certain quotient space by using a formula which relates the two operators L and L^*. In the remainder of this section we state this formula and refer the reader to Miranda [3] for additional information on this.

Let S be a C^1 hypersurface in Ω. If $\vec{n}(x) = (n_1(x),...,n_m(x))$ is a continuous vector field normal to S, then by setting

$$a(x) = \left[\sum_{i=1}^{m} \left[\sum_{j=1}^{m} a_{ij}(x)n_j(x) \right]^2 \right]^{1/2} \neq 0 \ ,$$

we define a *conormal* to S as being a vector field $\vec{\nu}(x) = (\nu_1(x),...,\nu_m(x))$ on S with continuous components,

$$\nu_i(x) = \frac{1}{a(x)} \sum_{j=1}^{m} a_{ij}(x)n_j(x) \ ; \ i = 1,...m \ ,$$

and whose direction is never, by (1.2), tangent to S.

The directional derivative in the direction of a conormal to S will be,

$$\frac{\partial}{\partial \nu} \equiv \vec{\nu} \cdot \nabla = \frac{1}{a(x)} \sum_{i,\,j=1}^{m} a_{ij}(x)n_j(x) \frac{\partial}{\partial x_i} \ ,$$

and we shall call the differential operator $\partial / \partial \nu_L = a(x) \, \partial / \partial \nu$ a *conormal derivative*. When $a_{ij} = \delta_{ij}$ then $a = 1$ and $\vec{\nu} = \vec{n}$.

GREEN'S IDENTITY. Let D be a subdomain with compact closure in Ω and with a C^1 boundary ∂D. If $u,v \in C^2(\bar{D})$ then,

$$\int_D \{v(x)Lu(x) - u(x) L^* v(x)\}dx = \int_{\partial D} [v(x),u(x)] \ , \qquad (1.6)$$

with,

$$[v(x),u(x)] =$$

$$\{v(x) \frac{\partial u(x)}{\partial \nu_L} - u(x) \frac{\partial v(x)}{\partial \nu_L} + e(x)u(x)v(x)\}ds \ ,$$

where,

$$e(x) = \sum_{i=1}^{m} e_i(x)n_i(x),$$

$$e_i(x) = \beta_i(x) - \sum_{j=1}^{m} \partial a_{ij}(x)/\partial x_j, \ i=1,...,m$$

and where $n_i(x)$ are the components of the unit normal exterior to D.

If u or v in (1.6) has compact support then we obtain a formula which justifies the terminology used for L^* .

2.   SPACES OF GERMS AND STATEMENT OF THE DUALITY THEOREM.

In order to describe our dual space $L(\Omega)'$ we need to introduce now some terminology and to make some remarks.

We shall say that a subdomain D of Ω , relatively compact and having a C^∞ boundary, is a *normal* subdomain when $\Omega\backslash D$ does not have any compact connected component. An open subset U of Ω will be called a (punctured) *neighborhood of infinity* if there exists a normal subdomain D of Ω such that $\Omega\backslash\overline{D} \subset U$. If U is a neighborhood of infinity and if $u \in L^*(U)$, then we say that (u,U) (or simply u) is a *solution at infinity* of $L^*u = 0$.

Let U_1 and U_2 be two neighborhoods of infinity and let (u_1,U_1) and (u_2,U_2) be two solutions at infinity of $L^*u = 0$. Then we say that u_1 and u_2 are *equivalent* if there exists a neighborhood of infinity in $U_1 \cap U_2$ on which u_1 and u_2 coincide. The equivalence classes so obtained are called *germs* of solutions at infinity and we denote by $L^*(\infty)$ the linear space of germs of solutions at infinity of $L^*u = 0$. Here the symbol $\infty = \infty_\Omega$ represents the point such that $\Omega \cup \{\infty\}$ is the Alexandroff compactification of Ω . Therefore a germ [u] belongs to $L^*(\infty)$ if $L^*u = 0$ on a (punctured) neighborhood of ∞ in Ω .

Uniqueness properties for solutions of elliptic equations were studied in a large number of works and one can obtain several references about this in Miranda [3]. The statement that follows comes from Agmon [4, page 151].

UNIQUENESS THEOREM.   Let L be a second order linear partial differential operator of elliptic type on a domain Ω of $\mathbb{R}^m$ with coefficients in $C^\infty(\Omega)$. If a C^2 solution u of $Lu = 0$ has a zero of infinite order at a point in Ω then $u \equiv 0$.

In view of the uniqueness theorem we may identify a solution $u \in L^*(\Omega)$ with the germ $[u] \in L^*(\infty)$. The mapping from $L^*(\Omega)$ to $L^*(\infty)$ which maps each u to its equivalence class [u] is indeed injective because if $[u_1] = [u_2]$ with u_1 and u_2 in $L^*(\Omega)$ then $u_1 - u_2$ vanishes on a neighborhood of infinity and, therefore, identically on Ω by the uniqueness theorem.

Let W_1 and W_2 be two normal subdomains of Ω and let u and v be two functions of class C^2 satisfying $Lu = 0$ and $L^*v = 0$ on a neighborhood of $\Omega\backslash W_j$, $j = 1,2$. Then by applying Green's identity (1.6) on $W_0\backslash\overline{W}_j$ where W_0 is a normal subdomain such that $\overline{W}_1 \cup \overline{W}_2 \subset W_0$ we obtain, for $j = 1,2$,

$$\int_{\partial W_0} [v(x),u(x)] - \int_{\partial W_j} [v(x),u(x)] =$$

$$\int_{W_0\backslash\overline{W}_j} \{v(x)Lu(x) - u(x)L^*v(x)\}dx = 0 .$$

(2.1)

Formula (2.1) means that the integral $\int_{\partial W} [v,u]$ does not depend on the choice of the normal subdomain W such that $u \in L(\Omega \backslash W)$ and $v \in L^*(\Omega \backslash W)$. We can therefore define the integral on the ideal boundary of Ω (obtained by compactifying Ω in the sense of Alexandroff) as follows,

$$\int_\infty [v,u] = \lim_{w \to \Omega} \int_{\partial W} [v,u] , \qquad (2.2)$$

whenever u and v are solutions at infinity of $Lu = 0$ and $L^* v = 0$.

Finally, by choosing an equivalence class Θ in $L^*(\infty)$ and an element $v \in \Theta$ we see that the continuous linear functional defined by,

$$L(\Omega) \ni u \longrightarrow \int_\infty [v,u] , \qquad (2.3)$$

does not depend on the choice of $v \in \Theta$.

We can now state our result.

DUALITY THEOREM. There is a vector–space isomorphism

$$L(\Omega)' = L^*(\infty)/L^*(\Omega) \qquad (2.4)$$

where the action of a germ in $L^*(\infty)$ on a solution in $L(\Omega)$ is given by

$$(v,u) = \int_\infty [v,u] . \qquad (2.5)$$

A representation similar to (2.5) was obtained by M. Nakai, L. Sario [1] for harmonic functionals on noncompact Riemann surfaces. When the operator L is self adjoint, as it is in the harmonic case, then the isomorphism (2.4) reduces to the one of Nakai, Sario.

3.   SOME TOOLS.

In order to make easier the reading of the proof of the duality theorem, we present in this section, some of the results that will be used in the next sections.

The first result is an extension, to solutions of second order linear partial differential equations of elliptic type, of the classical Runge approximation property for harmonic functions. The statement that follows comes from Lax [5, page 760].

LAX'S EQUIVALENCE THEOREM. Let L be a second order linear partial differential operator of elliptic type with C^∞ coefficients in a domain Ω of $\mathbb{R}^m$ and let D be a normal subdomain of Ω. Every C^2 solution of $Lu = 0$ in D is the uniform limit on compact subsets of D of a sequence of solutions of $Lu = 0$ in Ω if and only if every solution of the adjoint equation $L^* u = 0$ vanishing on a hypersurface S and whose conormal derivative vanishes on S is identically zero.

The existence and the properties of fundamental solutions for elliptic operators will also be used. We now recall the main facts connected with them and, as in section 1, we refer the reader to Miranda [3] for additional information on the remainder of this section.

Let (a_{ij}) be the matrix of coefficients of the principal part of L. We denote by A_{ij} the elements of the inverse matrix of (a_{ij}) and by A the determinant of (a_{ij}). We set,

$$
H(x,y) = \begin{cases}
\dfrac{1}{(m-2)\omega_m \sqrt{A(y)}} \left[\sum_{i,\,j\,=\,1}^{m} A_{ij}(y)(x_i-y_i)(x_j-y_j) \right]^{(2-m)/2} \\[4pt]
\text{for } m > 2 , \\[20pt]
\dfrac{1}{2\pi \sqrt{A(y)}} \log \left[\sum_{i,\,j\,=\,1}^{m} A_{ij}(y)(x_i-y_i)(x_j-y_j) \right]^{-1/2} \\[4pt]
\text{for } m = 2 .
\end{cases}
$$

If $r = |x - y|$ then in every relatively compact subdomain of Ω we have by (1.3),

$$
H = O(r^{2-m}) \,,\; \frac{\partial H}{\partial x_i} = O(r^{1-m}) \,,\; \frac{\partial^2 H}{\partial x_i\, \partial x_j} = O(r^{-m}) \,.
$$

A function $E(x,y)$ will be called a *fundamental solution* of the equation $Lu = 0$ $(L^* u = 0)$ on a domain W if it satisfies the following properties:

i) $E(x,y)$ and all its partial derivatives of first and second order with respect to the coordinates x_i of the variable x are continuous on $W \times W$ outside the diagonal.

ii) $E(x,y)$ and all its partial derivatives of first and second order with respect to the coordinates x_i of the variable x satisfy, for some $\lambda > 0$, the following bounds,

$$
E-H = O(r^{\lambda+2-m}) \,,\; \frac{\partial(E-H)}{\partial x_i} = O(r^{\lambda+1-m}) \,,
$$

$$
\frac{\partial^2(E-H)}{\partial x_i\, \partial x_j} = O(r^{\lambda-m}) \,,
$$

uniformly on every relatively compact subdomain of W.

iii) $L_x E(x,y) = 0$ $(L_x^* E(x,y) = 0)$ for each $x \in W$ with $x \neq y$.

Any fundamental solution $E(x,y)$ on a domain W together with all its partial derivatives of first order with respect to the coordinates x_i, are locally integrable in each variable and also on

$W \times W$. Moreover if $E(x,y)$ is a fundamental solution of the equation $L^*u = 0$ then by [3, page 19, (9.3)] we have $L_x^* E(x,y) = \delta_y$. Therefore E is a fundamental solution in the sense of distribution theory.

GIRAUD'S THEOREM. [3, page 66]. Let Ω be a domain in $\mathbb{R}^m$ and let y be a point of Ω . Each C^2 solution u of $Lu = 0$ in $\Omega\backslash\{y\}$ satisfying,

$$u(x) = o(H(x,y)) \quad \text{when } x \longrightarrow y \ ,$$

extends to a solution of $Lu = o$ on the entire domain Ω .

4.   PROOF OF FORMULA 2.5.
     We set,

$$F(\Theta)(u) = \int_\Omega [v,u] \ ,$$

and obtain a linear mapping,

$$F : L^*(\infty) \quad \longrightarrow \quad L(\Omega)' \ . \tag{4.1}$$

Given $\ell \in L(\Omega)'$, we look for $\Theta \in L^*(\infty)$ such that $F(\Theta) = \ell$. We first extend ℓ to a continuous linear functional on $C^0(\Omega)$, [6, page 108]. Then, [1, page 154], there exists a signed regular Borel measure μ with compact support in Ω such that,

$$\ell(u) = \int ud\mu \tag{4.2}$$

for every $u \in L(\Omega)$.
     We denote by S_μ the intersection of all normal subdomains of Ω containing the support of μ . Then S_μ is a compact subset of Ω and contains the support of μ . Let $\{W_j\}_{j=1}^\infty$ be an exhaustion of Ω by normal subdomains such that $S_\mu \subset W_1$. For each integer $j \geq 1$, let $E_j(x,y)$ be a fundamental solution of the equation $L^*v = 0$ on W_j . Since L^* has its coefficients in $C^\infty(\Omega)$ and since W_j has compact closure in Ω , the existence of $E_j(x,y)$ is guaranteed, [3, page 64].

     For each $j \geq 1$, we define a continuous function $v_j(x)$ by setting,

$$v_j(x) = \int E_j(x,y)d\mu(y) \ , \tag{4.3}$$

for every $x \in W_j\backslash S_\mu$ with μ given by (4.2).

     Let $\varphi \in C_c^\infty(W_j\backslash S_\mu)$. Since $E_j(x,y)$ is integrable, we find by using Fubini's theorem and

Green's identity on a regular domain G such that $\operatorname{supp} \varphi \subset G \subset W_j \backslash S_\mu$,

$$\int_{W_j \backslash S_\mu} v_j(x) L\varphi(x) dx = 0 \ ,$$

and we conclude by Weyl's lemma that $v_j \in C^\infty(W_j \backslash S_\mu)$ and that $L^* v_j(x) = 0$ on $W_j \backslash S_\mu$.

Let D be a normal subdomain such that $S_\mu \subset D \subset \bar{D} \subset W_1$. If $u \in L(\Omega)$ then by Green's identity we have,

$$\int_{\partial D} [v_{j+1}(x), u(x)] = \int_{\partial W_j} [v_{j+1}(x), u(x)] \ , \tag{4.4}$$

for each $j \geq 1$ and,

$$u(y) = \int_{\partial W_j} [E_{j+1}(x,y), u(x)] \ , \tag{4.5}$$

for each $y \in W_j$, [3, page 19]. Therefore by Fubini's theorem and by formulas (4.3), (4.4), (4.5) and (4.2) we find,

$$\int_{\partial D} [v_{j+1}(x), u(x)] = \int_{\partial W_j} [\int E_{j+1}(x,y) d\mu(y), u(x)] =$$

$$\int \int_{\partial W_j} [E_{j+1}(x,y), u(x)] d\mu(y) = \int u(y) d\mu(y) = \ell(u) \ ,$$

and from this we conclude that,

$$\ell(u) = \int_{\partial D} [v_j(x), u(x)] \ , \tag{4.6}$$

for each $u \in L(\Omega)$ and each $j \geq 2$.

In order to complete the proof we wish to replace v_j in (4.6) by an element $v \in L^*(\Omega \backslash S_\mu)$.

We fix a positive integer t and consider for each positive integer k , functions defined by $V_k(x) = v_{t+k+1}(x) - v_{t+k}(x)$ on $W_{t+k} \backslash S_\mu$. By Giraud's theorem, the two fundamental solutions $E_{t+k+1}(x,y)$ and $E_{t+k}(x,y)$ differ at most on W_{t+k} by a function $g_k(x,y)$ of class C^2 in the variable x satisfying $L_x^* g_k(x,y) = 0$ on W_{t+k} for each $y \in W_{t+k}$.

We claim that $g_k(x,y)$ is continuous on $W_{t+k} \times W_{t+k}$. To prove this we write L^* as,

$$L^* = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i} + b(x) \,,$$

where $b_i(x)$ and $b(x)$ are of class C^∞. Since $g_k(x,y)$ is continuous on $W_{t+k} \times W_{t+k}$ when $x \neq y$, it suffices to prove the continuity of $g_k(x,y)$ at those points on the diagonal.

We show that $g_k(x,y)$ is continuous on $B \times B$ where B is any small ball in W_{t+k}. Let us consider such a ball B sufficiently small to guaranty the existence, in a neighborhood ω of B, of a positive C^2 solution $w(x)$ of $L^* w(x) = -1$, [3, page 65 and 66].

For each $x \in \omega$ and each $y \in W_{t+k}$ we set,

$$g_k(x,y) = \frac{g_k(x,y)}{w(x)} \, w(x) \equiv g_k^1(x,y)w(x) \,,$$

and we show that $g_k^1(x,y)$ is continuous on $B \times B$. Let us remark that $g_k^1(x,y)$ satisfies an elliptic equation $A_x g_k^1(x,y) = 0$ on ω where

$$A = \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} +$$

$$\sum_{i=1}^{m} \left[b_i(x) + \frac{2}{w(x)} \sum_{j=1}^{m} a_{ij}(x) \frac{\partial w(x)}{\partial x_j} \right] \frac{\partial}{\partial x_i} - \frac{1}{w(x)} \,.$$

We choose a point (x_1, y_1) in $B \times B$ and write,

$$|g_k^1(x,y) - g_k^1(x_1,y_1)| \leq |g_k^1(x,y) - g_k^1(x,y_1)| +$$

$$|g_k^1(x,y_1) - g_k^1(x_1,y_1)| \,. \tag{4.7}$$

By the continuity of $g_k^1(x,y_1)$ at the point x_1, we see that the second term in the right member of inequality (4.7) can be made arbitrarily small when x is close to x_1. By the maximum principle [3, page 7],

$$|g_k^1(x,y) - g_k^1(x,y_1)| \leq \sup_{\partial B} |g_k^1(\cdot,y) - g_k^1(\cdot,y_1)| \,, \tag{4.8}$$

for each x and y in B. But by the continuity of $g_k^1(x,y) - g_k^1(x,y_1)$ outside the diagonal, the right member of (4.8) and therefore those of (4.7) can be made arbitrarily small when y

approaches y_1 and (x,y) approaches (x_1,y_1). This proves the continuity of $g_k^1(x,y)$ at (x_1,y_1) and therefore in $B \times B$. Thus $g_k(x,y)$ is continuous on $B \times B$ and therefore on $W_{t+k} \times W_{t+k}$. This proves the claim.

Using the claim, we extend continuously $V_k(x)$ by setting,

$$V_k(x) = \int g_k(x,y)d\mu(y) ,$$

for each $x \in W_{t+k}$. Using Fubini's theorem and Green's identity as above we obtain,

$$\int\limits_{W_{t+k}} V_k(x)L\varphi(x)dx = 0 ,$$

for every $\varphi \in C_c^\infty(W_{t+k})$ and we conclude by Weyl's lemma that $V_k \in L^*(W_{t+k})$.

By the uniqueness theorem we may apply Lax's equivalence theorem in order to obtain, for each positive integer k, a global solution $h_k \in L^*(\Omega)$ such that,

$$\sup\left\{ |V_k(x) - h_k(x)| : x \in \overline{W}_{t+k-1} \right\} < 1/2^k . \tag{4.9}$$

Estimate (4.9) shows that the series $\sum_{k=1}^\infty (V_k - h_k)$ converges uniformly on compact subsets of Ω.

Let us consider the series defined on $W_{t+1} \backslash S_\mu$ by,

$$v = v_{t+1} + \sum_{k=1}^\infty (V_k - h_k) ,$$

and whose sum belongs to $L^*(W_{t+1} \backslash S_\mu)$. Since $v_{t+\ell} + V_\ell = v_{t+\ell+1}$ on $W_{t+\ell} \backslash S_\mu$ for each $\ell = 1,...,k-1$ and each $k \geq 2$, we see that

$$v_{t+1} + \sum_{i=1}^{k-1} V_i = v_{t+k} ,$$

on $W_{t+1} \backslash S_\mu$. This shows that, for each positive integer k, the function defined on $W_{t+k} \backslash S_\mu$ by the series

$$v_{t+k} - (h_1 + ... + h_{k-1}) + \sum_{\ell=k}^\infty (V_\ell - h_\ell) ,$$

extends v to an element of $L^*(W_{t+k}\setminus S_\mu)$. Such an extension being possible for every k , we see that v extends to an element, still denoted by v , of $L^*(\Omega\setminus S_\mu)$.

Thus by applying Green's identity on a subdomain D such that $S_\mu \subset D \subset \bar{D} \subset W_1$ to an element $u \in L(\Omega)$ and by using (4.6) we have,

$$\int_{\partial D} [v(x),u(x)] =$$

$$\int_{\partial D} [v_{t+1}(x),u(x)] + \int_{\partial D} [\sum_{k=1}^{\infty} (V_k(x) - h_k(x)),u(x)] =$$

$$= \ell(u) + \int_{D} \left\{ \sum_{k=1}^{\infty} (V_k(x) - h_k(x))Lu(x) - u(x)L^* \sum_{k=1}^{\infty} (V_k(x) - h_k(x)) \right\} dx .$$

Therefore,

$$\ell(u) = \int_{\partial D} [v(x),u(x)] ,$$

for every $u \in L(\Omega)$. Since $v \in L^*(\Omega\setminus S_\mu)$ we obtain by (2.1) the desired representation,

$$\ell(u) = \int_{\omega} [v(x),u(x)] ,$$

and the existence of an element $\Theta = [v] \in L^*(\omega)$ such that $F(\Theta) = \ell$.

5.   PROOF OF ISOMORPHISM 2.4.

Let us denote by $\ker F$ the kernel of the mapping (4.1). In order to complete the proof of the duality theorem, we have to show that $\ker F = L^*(\Omega)$.

It is clear that $\ker F \supset L^*(\Omega)$. Indeed if $v \in L^*(\Omega)$ then for every $u \in L(\Omega)$ we have,

$$F(\Theta)(u) = \int_{\omega} [v,u] = \lim_{w\to\Omega} \int_{\partial W} [v,u] =$$

$$\lim_{w\to\Omega} \int_{W} \{vLu - uL^*v\}dx = 0 .$$

Conversely let us take an element $v = (v,U) \in \Theta \in \ker F$ and a normal subdomain W such that $\Omega\setminus W \subset U$. We extend v to an element $\tilde{v}$ in $C^2(\Omega)$ and we consider $g \in C^2(\bar{W})$ such that $Lg = 0$ in W and $g = 0$ on ∂W .

By Green's identity we have,

$$\int_{\partial W} v \frac{\partial g}{\partial \nu_L} ds = - \int_{W} gL^* \tilde{v}dx . \tag{5.1}$$

The right integral in (5.1) is taken on the support of $L^*\tilde{v}$ which is a compact subset of W. By Lax's equivalence theorem, $L(\Omega)$ is dense in $L(W)$ and therefore there exists a sequence $\{g_j\}_{j=1}^{\infty}$ in $L(\Omega)$ such that $\lim_{j\to\infty} g_j = g$ uniformly on the support of $L^*\tilde{v}$. Since $v \in \Theta \in \ker F$ we have by Green's identity,

$$- \int_W g L^*\tilde{v}dx = -\lim_{j\to\infty} \int_W g_j L^*\tilde{v}dx = \lim_{j\to\infty} \int_{\partial W} [v, g_j] = 0 \, ,$$

and (5.1) becomes,

$$\int_{\partial W} v \frac{\partial g}{\partial \nu_L} ds = 0 \, . \tag{5.2}$$

Therefore, [3, pages 76, 77 and 78], (5.2) implies the existence of $h \in C^2(\overline{W})$ such that $L^*h = 0$ in W and $h = v$ on ∂W. We are going to show that there exists a function h_1 satisfying these properties and such that $\partial(v-h_1)/\partial\nu_L = 0$ on ∂W.

By Green's identity we have,

$$\int_W u L^*\tilde{v}dx = \int_{\partial W} u \frac{\partial}{\partial \nu_L}(v-h)ds \, , \tag{5.3}$$

for every C^2 solution of $Lu = 0$ in W taking boundary values of class C^2 on ∂W. By an approximation theorem of Walsh–Mergelyan type for solutions of $Lu = 0$ [7, page 150, theorem 2], the formula (5.3) remains valid for any solutions of $Lu = 0$ in W taking continuous boundary values on ∂W. Therefore, by Lax's equivalence theorem, formula (5.3) becomes,

$$\int_{\partial W} u \frac{\partial}{\partial \nu_L}(v-h)ds = 0 \, , \tag{5.4}$$

for every $u \in L(W) \cap C^0(\overline{W})$.

Let us denote by P the prehilbert space $L^2(\partial W, ds) \cap C^0(\partial W)$, by Σ the subspace of the boundary values of functions $u \in C^0(\overline{W})$ such that $Lu = 0$ in W and by Φ the subspace of P of all functions $\partial\varphi/\partial\nu_L$ where $\varphi \in C^2(\overline{W})$ satisfies $L^*\varphi = 0$ in W and $\varphi = 0$ on ∂W. The space of such φ has finite dimension, [3, pages 76 and 77]. Moreover if $\{\varphi_i\}_{i=1}^k$ is a basis of this space then $\{\partial\varphi_i/\partial\nu_L\}_{i=1}^k$ is a finite basis of Φ. Therefore, by a property of finite dimensional linear subspaces of a prehilbert space, [8, page 66, theorem 3], we have $\Phi^{\perp\perp} = \Phi$. But since we already know that $\Phi^{\perp} = \Sigma$, [3, pages 77 and 78], we conclude that $\Phi = \Sigma^{\perp}$.

By (5.4), $\partial(v-h)/\partial\nu_L \in \Sigma^{\perp}$. Therefore, there exists $h_0 \in C^2(\overline{W})$ satisfying $L^*h_0 = 0$ in W, $h_0 = 0$ on ∂W and $\partial(v-h)/\partial\nu_L = \partial h_0/\partial\nu_L$ on ∂W. By setting $h_1 = h + h_0$, we obtain an element of $C^2(\overline{W})$ satisfying $L^*h_1 = 0$ in W and $v-h_1 = \partial(v-h_1)/\partial\nu_L = 0$ on ∂W.

Let $\hat{v}$ be the function on $\Omega \backslash \partial W$ which coincide with v on $\Omega \backslash \overline{W}$ and with h_1 on W Then by the preceding remark $\hat{v}$ extends continuously to the entire domain Ω and satisfies the following relation,

$$\int_{\Omega} \hat{v}(x) L\varphi(x) dx = 0 , \qquad (5.5)$$

for every $\varphi \in C_c^{\infty}(\Omega)$. To prove (5.5) let φ be in $C_c^{\infty}(\Omega)$ and let D be a domain with compact closure in Ω containing the support of φ such that the two subdomains $D_1 = D \cap (\Omega \backslash \overline{W})$, $D_2 = D \cap W$ have a boundary of class C^1. By using the fact that $v - h_1$ and its conormal derivative vanish on ∂W, we see that the integral in (5.5) is equal to

$$\int_{D_1} \varphi(x) L^* v(x) dx + \int_{D_2} \varphi(x) L^* h_1(x) dx ,$$

and (5.5) follows. Therefore, by Weyl's lemma $\hat{v} \in L^*(\Omega)$. Thus $[v] = [\hat{v}] \in L^*(\Omega)$ and ker $F \subset L^*(\Omega)$. This proves the duality theorem.
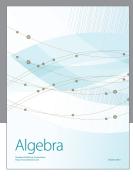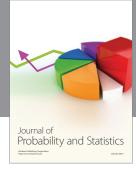
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