ON SEMI-HOMEOMORPHISMS

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ABSTRACT. In the first part of our work we show a condition for a semi-homeomorphism in the sense of Crossley and Hildebrand (s.h.C.H) to be a semi-homeomorphism in the sense of Biswas (s.h.B). Certain relevant examples are provided. Next, we define strong semi-homeomorphisms via "nice" restrictions of semi-homeomorphisms ("global condition") and we show that the new class of functions actually coincides with semi-homeomorphisms. Then, in the third part we introduce local semi-homeomorphisms (l.s.h.C.H.) via a corresponding "local condition" for restrictions. A few results pertaining to the preservation of some topological properties under this new class of functions are examined.

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1. s.h.C.H. VERSUS s.h.B.

We shall start with the following definitions.

A subset $S \subseteq X$ is said to be $\text{semi-open}$ if there is an open set $U \subseteq X$ such that $U \subseteq S \subseteq \bar{U}$.

A function $f : X \rightarrow Y$ is said to be a $\text{semi-homeomorphism in the sense of Crossley and Hildebrand}$ (or simply, s.h.C.H.) [1] if:

1. $f$ is bijective
2. $f$ is irresolute (i.e. inverse images of semi-open sets are semi-open)
3. $f$ is pre-semi-open (i.e. images of semi-open sets are semi-open)

Further, a function $f : X \rightarrow Y$ is said to be a $\text{semi-homeomorphism in the sense of Biswas}$ (or simply s.h.B. [2]), if
1. f is bijective
2. f is continuous
3. f is semi-open

Clearly every homeomorphism is both s.h.B and s.h.C.H.

T. Neubrunn [3] has shown that there are s.h.C.H. that are not s.h.B. Answering his question Z. Piotrowski [4] has shown an example of a s.h.B. which is not s.h.C.H. Further he also obtained certain conditions for a s.h.C.H. to be a s.h.B.

In this paragraph we shall prove the following:

Proposition 1. Assume Y has a clopen base. If f: X→Y is one-to-one, semi-open and somewhat continuous then f is irresolute.

PROOF: Let A ⊆ Y be semi-open. Let x ∈ f^(-1)(A), i.e., f(x) ∈ Y ⊆ A. We shall show that x ∈ Int f^(-1)(A).

For any open set U containing x, the set f(U) is semi-open and contains y. Further f(U) ∩ A ≠ ∅. Since f(U) is open, Y having a clopen base and A is open - all semi-open and open sets coincide, under the assumption upon Y, there is a nonempty open set G such that

G = f(U) ∩ A. \hspace{1cm} (1.1)

Clearly,

f^(-1)(G) = f^(-1)(f(U) ∩ A) ⊆ f^(-1)(f(U)) = U \hspace{1cm} (1.2)

f being one-to-one.

Now, somewhat continuity of f implies that there is an open set V ⊆ f^(-1)(G), V ≠ ∅. Therefore V ⊆ U, V ⊆ f^(-1)(A). And since U is an arbitrary neighborhood of x, we have x ∈ Int f^(-1)(A). Thus f^(-1)(A) is semi-open.

REMARK: The author is indebted to the referee for pointing out that Proposition 1 generalizes Theorem 2.2 of [5].

The assumption upon Y to have a clopen base is essential. In fact:

EXAMPLE 2. There is a semi-open, semi-continuous (hence somewhat continuous!) bijection f: [0,1] → [0,1] which is not irresolute. Take f(x) = x, if x ∈ [0,\frac{1}{3}], f(x) = x + \frac{1}{3}, if x ∈ [\frac{1}{3}, \frac{2}{3}], f(x) = -x + \frac{4}{3}, if x ∈ [\frac{2}{3}, 1]. Observe that f^(-1)[\frac{1}{3}, \frac{2}{3}] is not semi-open.

In fact, there is even a continuous, semi-open injective function between two topological spaces which is not irresolute. We shall provide here such an example, originally designed for a different purpose.

EXAMPLE 3. ([4], Example 19, p. 8) Let X = Y = {a, b, c, d}. Let Ω_1 and Ω_2 denote the topologies for X and Y, respectively, such that Ω_1 = {∅, X, {a}, {b}, {a,b}, {b,c,d}} and Ω_2 = {∅, Y, {a}, {b}, {a,b}}. Let f:(X,Ω_1) → (Y,Ω_2) be the identity function. It is easy to see that f is continuous and semi-open but not irresolute, since {a,c} is semi-open in Y while it is not semi-open in X.

REMARK 4. Example 2 above is the best possible in the class of semi-continuous bijections f: [0,1] → [0,1] (or more generally, f: X→Y, X-compact, Hausdorff and Y
being Hausdorff, in the sense that if \( f \) is to be additionally continuous, then, being continuous bijection from a compact, Hausdorff space onto a Hausdorff space, it is a homeomorphism, (see [6], Thm 2.1, p. 226). Now, every homeomorphism (actually openness and continuity suffices) implies irresoluteness of \( f \) - we leave the proof of this fact to the reader, also see [1].

Since it is well-known that continuity and somewhat openness imply pre-semi-openness, see also [1] we have the following Corollary from Proposition 1.

**COROLLARY 5.** Assume \( Y \) has a clopen base. If \( f: X \to Y \) is s.h.B, then \( f \) is s.h.C.H.

2. STRONG SEMI-HOMEOMORPHISMS ARE PRECISELY SEMI-HOMEOMORPHISMS.

In this paragraph a "semi-homeomorphism" stands for s.h.C.H.

The following, seemingly stronger conditions (*) and (***) which define what we call a *strong semi-homeomorphism* are actually equivalent (!) to the semi-homeomorphicity of \( f \), see the following.

**THEOREM 6.** A function \( f: X \to Y \) is a semi-homeomorphism if and only if:

(*) \( f \) is bijective and

(***) \( \forall U \subseteq X, U \text{-open, } f|_U \text{ is a semi-homeomorphism} \)

**PROOF:** In fact, it is easy to see that if \( f \) satisfies (*) and (***) then \( f \) is a semi-homeomorphism - take \( U = X \) in (**). Conversely, let \( X = U \cup U_\alpha : \alpha \in \mathcal{A} \), where each \( U_\alpha \) is open and suppose that each restriction \( f|_{U_\alpha} \) is both pre-semi-open and irresolute. We shall show that \( f \) is also such.

Let \( (f|_{U_\alpha})_\alpha \) denote the restriction of \( f \) to \( U_\alpha \). We shall show that \( f \) is irresolute. Really, given a semi-open set \( K \subseteq Y \) we have:

\[
 f^{-1}(K) = U \{ f^{-1}(K) \cap U_\alpha : \alpha \in \mathcal{A} \} = U \{(f|_{U_\alpha})^{-1}(K) : \alpha \in \mathcal{A} \}. \tag{2.1}
\]

The latter set is semi-open as the sum of semi-open sets.

Similarly, we shall prove that \( f \) is pre-semi-open. Let \( L \subseteq X \) be semi-open, in \( X \). Then \( L = U \{ L \cap U_\alpha : \alpha \in \mathcal{A} \} \). Then:

\[
 f(L) = f(U \{ L \cap U_\alpha : \alpha \in \mathcal{A} \}) =
 = U \{ f(L \cap U_\alpha : \alpha \in \mathcal{A} \} =
 = U \{(f|_{U_\alpha})(L) : \alpha \in \mathcal{A} \}. \tag{2.2}
\]

And again, the latter set is semi-open, in \( Y \).

The following example shows that the assumption "for every open" in (***) is real. As one can see, the restrictions \( f|_{U_\alpha} \), \( \alpha \in \mathcal{A} \) are even homeomorphisms (!) for every \( U_\alpha \nsubseteq X \).

**EXAMPLE 7.** (See Example 3 of §1.) There is a function \( f: X \to Y \) such that

1. \( f \) is bijective and
2. \( \forall U_\alpha \subseteq X, U_\alpha \text{-open, } \alpha \in \mathcal{A}, f|_{U_\alpha} \text{ is a homeomorphism} \)
(hence, a semi-homeomorphism) whereas \( f: X \rightarrow Y \) is not a semi-homeomorphism.

Really, \( f(a) \), \( f(b) \) and \( f(a,b) \) are homeomorphisms. Now, consider \( f(b,c,d) \). We have \( X = Y = \{b,c,d\} \) and \( 0_1 = X \setminus \{\emptyset, \{b,c,d\}, \{b\}\} \), whereas \( 0_2 = Y \setminus \{\emptyset, \{b,c,d\}, \{b\}\} \). And, here again, \( f(b,c,d) \) is a homeomorphism.

3. LOCAL SEMI-HOMEOMORPHISMS.

Local homeomorphisms, being a very natural generalization of homeomorphisms, occupy an important place in topology, especially in the theory of 1-dimensional continua (curves) as well as some parts of algebraic topology, see also [7] for an extensive treatment of this topic.

Let us define our new class of functions. We say that a function \( f: X \rightarrow Y \) is a local semi-homeomorphism in the sense of Crossley and Hildebrand if:

1. \( f \) is bijective and
2. \( \forall x \in X \exists U \text{-open}, x \in U \subseteq X \) such that \( f|_U \) is a semi-homeomorphism in the sense of Crossley and Hildebrand.

Well, it is easy to see that every semi-homeomorphism is a local semi-homeomorphism; take \( U = X \). Since every homeomorphism is a semi-homeomorphism, see [1] we have the following diagram:

\[
\text{strong homeomorphism} \rightarrow \text{semi-homeomorphism} \leftrightarrow \text{semi-homeomorphism} \rightarrow \text{local semi-homeomorphism}
\]

We shall now provide an example of a local semi-homeomorphism which is not a semi-homeomorphism, showing that the arrow to the right is, in general, not reversible.

EXAMPLE 8. Consider Example 3, see §1. Take \( \{a\}, \{b\}, \{b,c,d\}, \{b,c,d\} \), respectively for open neighborhoods of \( a, b, c \) and \( d \), respectively. Using arguments similar to ones applied in Example 7 we prove that \( f \) is a local semi-homeomorphism; it has been shown in [4], p. 508 that \( f \) is not a semi-homeomorphism.

LEMMA 9. If for every \( x \in X \) there is an open set \( U \subseteq X \), \( x \in U \) such that \( f|_U \) is a semi-homeomorphism in the sense of Crossley and Hildebrand, then \( f \) is somewhat continuous (inverse images of every nonempty open set if nonempty it has the nonempty interior) and \( f \) is somewhat open (image of every open nonempty set has the nonempty interior).

PROOF: Let \( D \) be a dense set in \( X \). We shall show that \( f(D) \) is dense in \( f(X) \). This, in turn, shows that \( f \) is somewhat continuous.

In fact, suppose \( y \in f(X) \setminus f(D) \) and assume further that there is an open neighborhood \( V \) containing \( y \), such that:

\[
(*) \quad V \cap f(D) = \emptyset. \tag{3.1}
\]

Since \( f \) is "onto", there is \( x \in X \), such that \( f(x) = y \). There is an open set \( U \ni x \) such that \( f|_U \) is a semi-homeomorphism, \( f \) being a local semi-homeomorphism. Clearly
D ∩ U is dense in U; further f(D ∩ U) is dense in f(U), f being semi-homeomorphism on U. Now, f(U) is a semi-open set containing f(x) = y. By an elementary property of semi-open sets, f(D ∩ U) is dense in V ∩ Int f(U), and hence, also in V ∩ f(U). So, V ∩ f(D) ≠ ∅, contradicting (**).

Now, for somewhat openness part, consider a dense set D contained in f(X). We shall show that f⁻¹(D) is dense (in X). Suppose f⁻¹(D) is not dense. So, there is a point x ∈ X and an open neighborhood U of x such that

(**) U ∩ f⁻¹(D) ≠ ∅.

Without loss of generality we may assume that U is the open neighborhood of x from the definition of local homeomorphism (or, simply, take the intersection of the two sets, in question). Then f(U) is a semi-open set, free of points of D. For otherwise the set:

\[ f⁻¹(f(U) ∩ D) = f⁻¹(f(U)) ∩ f⁻¹(D) \cap f⁻¹(D) \cap U \neq ∅, \]

contradicting (**), which finishes the proof.

COROLLARY 10. Baireness is a local semi-topological property.

PROOF: See [8], Corollary 2, p. 410 and Lemma 9, above.

COROLLARY 11. Separability is a local semi-topological property.

PROOF: Every local semi-homeomorphism is somewhat continuous, and this implies (see [9]) that dense subsets are preserved, which in turn proves our claim.

We will close this work with the following natural

Question 12. What are the topological conditions for X and/or Y so that every local semi-homeomorphism f: X → Y is a semi-homeomorphism?

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REFERENCES


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