ON CERTAIN REAL HYPERSURFACES OF QUATERNIONIC PROJECTIVE SPACE

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ABSTRACT. We classify certain real hypersurfaces of a quaternionic projective space satisfying the condition \( \sigma(R(X,Y)SZ) = 0 \).

KEY WORDS AND PHRASES: Quaternionic projective space, real hypersurface.


1. INTRODUCTION.

Let \( M \) be a connected real hypersurface of a quaternionic projective space \( QP^n, n \geq 2 \), with metric \( g \) of constant quaternionic sectional curvature \( 4 \). Let \( \xi \) be the unit local normal vector field on \( M \) and \( \{\psi_1, \psi_2, \psi_3\} \) a local basis of the quaternionic structure of \( QP^n \), (See [1]). Then \( U_i = \psi_i \xi, i=1,2,3 \) are tangent to \( M \). It is known, [3], that the unique Einstein real hypersurfaces of \( QP^n \) are the open subsets of geodesic hyperspheres of \( QP^n \) of radius \( r \) such that \( \cot^2 r = 1/(2n) \). This paper is devoted to study real hypersurfaces \( M \) of \( QP^n \) satisfying the following condition

\[ R(X,Y)SZ + R(Y,Z)SX + R(Z,X)SY = 0 \]  

(1.1)

for any \( X,Y,Z \) tangent to \( M \), where \( R \) denotes the curvature tensor and \( S \) the Ricci tensor of \( M \). Concretely we prove the following:

THEOREM 1. Let \( M \) be a real hypersurface of \( QP^n, n \geq 2 \), satisfying Condition (1.1) and such that \( U_i, i=1,2,3 \), are principal. Then \( M \) is an open subset of a geodesic hypersphere of \( QP^n \) of radius \( r, 0 < r < \pi/2 \), such that \( \cot^2 r = 1/(2n) \).

Clearly condition (1.1) is weaker than \( R.S=0 \). Thus we also obtain

COROLLARY 2. The unique real hypersurfaces of \( QP^n, n \geq 2 \), satisfying \( R.S=0 \) and such that \( U_i, i=1,2,3 \), are principal are open subsets of geodesic hyperspheres of radius \( r, 0 < r < \pi/2 \), such that \( \cot^2 r = 1/(2n) \).

COROLLARY 3. A real hypersurface of \( QP^n, n \geq 2 \), with \( U_i, i=1,2,3 \), principal cannot satisfy the condition \( R.R=0 \).

Where for any \( X,Y \) tangent to \( M \), \( R(X,Y)T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X,Y]} T \) for any tensor field \( T \) on \( M \), (see, for example, [5]).

2. PRELIMINARIES

Let \( X \) be a vector field tangent to \( M \). We write \( \psi_iX = \phi_iX + f_i(X)\xi, i=1,2,3 \), where \( \phi_iX \) denotes the tangential component of \( \psi_iX \) and \( f_i(X) = g(X, U_i) \). From this, [4], we have

\[ g(\phi_iX, Y) + g(X, \phi_iY) = 0, \quad \phi_iU_i = 0, \quad \phi_jU_k = -\phi_kU_j = U_i \]  

(2.1)
for any \( X \) and \( Y \) tangent to \( M \), \( i=1,2,3 \) and \( (j,k,t) \) being a cyclic permutation of \((1,2,3)\).

From the expression of the curvature tensor of \( QP^n \), [4], the equations of Gauss and Codazzi are given respectively by

\[
R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + \sum_{i=1}^{3} \{ g(\varphi_i Y, Z) \varphi_i X - g(\varphi_i X, Z) \varphi_i Y + 2g(\varphi_i X, Y) \varphi_i Z \} + g(AX, Z)AX - g(AX, Z)AY
\]

and

\[
(\nabla_X A)Y - (\nabla_Y A)X = \sum_{i=1}^{3} \{ f_i(X) \varphi_i Y - f_i(Y) \varphi_i X + 2g(X, \varphi_i Y)U_i \}
\]  

(2.2)  

(2.3)

for any \( X, Y, Z \) tangent to \( M \), where \( A \) denotes the Weingarten endomorphism of \( M \). The Ricci tensor of \( M \) has the following expression

\[
SX = (4n + 7)X - 3 \sum_{i=1}^{3} \{ f_i(X)U_i + hA X - A^2 X \}
\]  

(2.4)

for any \( X \) tangent to \( M \), \( h \) being the trace of \( A \).

If \( U_i, i=1,2,3 \), are principal and have the same principal curvature \( \alpha_i \), this is constant, [4], and from (2.3) it is easy to see that

\[
2A\varphi AX = \alpha_i(A\varphi_i + \varphi_i A)X + 2\varphi_i X + 2f_i(X)U_j - 2f_j(X)U_k
\]  

(2.5)

for any \( X \) tangent to \( M \), where \( (i,j,k) \) is a cyclic permutation of \((1,2,3)\).

3. PROOF OF THEOREM 1.

Let \( Z \) be a tangent field to \( M \), orthogonal to \( U_i, i=1,2,3 \), and principal with principal curvature \( \lambda \). Then, from Condition (1.1) and (2.4) we have

\[
(4n + 7 + hA \lambda - \lambda^2)R(U_1, U_2)Z + (4n + 4 + h\alpha_1 - \alpha_1^2)R(U_2, Z)U_1 + (4n + 4 + h\alpha_2 - \alpha_2^2)R(Z, U_1)U_2 = 0
\]  

(3.1)

where \( \alpha_i \) is the principal curvature of \( U_i, i=1,2,3 \).

From (3.1) and the identity of Bianchi we obtain

\[
(3 + h\lambda - \lambda^2)R(U_1, U_2)Z + (\alpha_1 - \alpha_1^2)R(U_2, Z)U_1 + (\alpha_2 - \alpha_2^2)R(Z, U_1)U_2 = 0
\]  

(3.2)

that is,

\[
(3 + h\lambda - \lambda^2 - h\alpha_1 + \alpha_1^2)R(U_1, U_2)Z + (h\alpha_2 - \alpha_2^2 - h\alpha_1 + \alpha_1^2)R(Z, U_1)U_2 = 0
\]  

(3.3)

From (2.2), (3.3) gives \( h\alpha_2 - \alpha_2^2 - h\alpha_1 + \alpha_1^2 = 2(3 + h\lambda - \lambda^2 - h\alpha_1 + \alpha_1^2) \). Changing \( (U_1, U_2) \) in (3.1) by \( (U_2, U_3) \) or \( (U_3, U_1) \), respectively, we obtain

\[
h\alpha_i - \alpha_i^2 + h\alpha_j - \alpha_j^2 = 6 + 2h\lambda - 2\lambda^2, i \neq i, j = 1, 2, 3
\]  

(3.4)

From (3.4) we get

\[
h(\alpha_i - \alpha_j) = \alpha_i^2 - \alpha_j^2
\]  

(3.5)

thus either \( \alpha_i = \alpha_j \) or \( \alpha_i + \alpha_j = h \).

Let us suppose that \( \alpha_1 \neq \alpha_2 = \alpha_3 \). Then \( \alpha_1 + \alpha_2 = h \). Thus \( \alpha_i, i=1,2,3 \), must satisfy the equation \( \alpha^2 - h\alpha + \alpha_1 \alpha_2 = 0 \). Then we have \( hA - A^2 U_i = \alpha_i \alpha_2 U_i, i=1,2,3 \), and from (2.4)

\[
SU_i = (4n + 4 + \alpha_1 \alpha_2)U_i
\]  

(3.6)
From (3.4) we also have $h(a_1+α_2)-α_1^2-α_2^2 = 6 + 2hα - 2λ^2$, but $h = α_1 + α_2$. Thus $α_1α_2 = 3 + hλ - λ^2$. This means that for any $Z$ orthogonal to $U_i$, $i=1,2,3$, $(hA-A^2)Z = (α_1α_2 - 3)Z$, and from (2.4),

$$SZ = (4n + 4 + α_1α_2)Z \tag{3.7}$$

From (3.6) and (3.7), $M$ must be Einstein. But this is a contradiction (see [3]). Thus $α_i = α_j = α$, $i \neq j$. Then $α$ is constant and from (3.4) we have

$$3 + h(λ - α) - λ^2 + α^2 = 0 \tag{3.8}$$

But from (2.5), $α_i Z$ is also principal and its principal curvature is $μ = (λα + 2)/(2λ - α)$. Thus we also get

$$3 + h(μ - α) - μ^2 + α^2 = 0 \tag{3.9}$$

Then from (3.8) and (3.9) we obtain that either $λ = μ$ or $λ + μ = h$. If $λ = μ$, $λ$ must satisfy the equation $λ^2 - λα - 1 = 0$. If $λ + μ = h$, $λ$ must satisfy the equation $αλ^2 - 2(α^2 + 4)λ + α^3 + 5α = 0$. In both cases all the principal curvatures are constant. Thus, [3], $M$ must be an open subset of either a geodesic hypersphere or of a tube of radius $r$, $0 < r < π/2$ over $QP^k$, $0 < k < n-1$. It is easy now to see that the only ones satisfying (3.8) are open subsets of geodesic hyperspheres of radius $r$, $0 < r < π/2$, such that $\cot^2 r = 1/(2n)$, (see [3]). This concludes the proof.

It is also easy to see that these real hypersurfaces cannot satisfy the condition $R.R=0$, and then Corollary 3 is proved because $R.R=0$ implies $R.S=0$.

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