A TRIGONOMETRICAL IDENTITY

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1. INTRODUCTION. The object of this note is to establish the following identity:

\[
\left\{ \frac{1}{4} \cot \left( \frac{\theta}{2} \right) + \sum_{k=1}^{\infty} \frac{x^{2k} \sin k\theta}{1 - x^{2k}} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{x^k \sin \left( k\theta/2 \right)}{1 - x^{2k}} \right\}^2
\]

\[
= \left\{ \frac{1}{4} \cot \left( \frac{\theta}{2} \right) \right\}^2 - \frac{3}{4} \sum_{k=1}^{\infty} \frac{x^{2k} \cos k\theta}{(1 - x^{2k})^2} + \frac{1}{8} \sum_{k=1}^{\infty} \frac{kx^{2k}}{1 - x^{2k}} (3 - 4 \cos k\theta)
\]

\[
+ \frac{1}{8} \sum_{k=1}^{\infty} \frac{kx^k \cos \left( k\theta/2 \right)}{1 - x^{2k}} - \frac{3}{8} \sum_{k=1}^{\infty} \frac{x^k (1 + x^{2k})}{(1 - x^{2k})^2} \cos \left( k\theta/2 \right),
\] (1.1)

valid for \( \theta \in \mathbb{R} \), \( x \in \mathbb{C} \), \( \theta \) not an even multiple of \( \pi \) and \( |x| < 1 \). The details of the proof are supplied in section 2. In our concluding remarks we compare (1.1) with a celebrated identity of Ramanujan, and discuss a uniform method which reveals a total of four such trigonometrical identities.

2. PROOF OF IDENTITY (1.1). Our argument is based on the following variant of the quintuple-product identity:

\[
\prod_{n=1}^{\infty} \frac{(1 - x^{2n}) (1 - a^{-2} x^{2n-2}) (1 - a^{-2} x^{2n})}{(1 + ax^{2n-1}) (1 + a^{-1} x^{2n-1})} = \sum_{n=-\infty}^{\infty} x^{n(3n+2)} (a^{-3n} - a^{3n} + 2),
\] (1.2)

valid for \( a, x \in \mathbb{C} \), \( a \neq 0 \) and \( |x| < 1 \). For a discussion of (1.2) and other forms of the quintuple-product identity see [1].

In (1.2) let \( a \rightarrow -a \), \( x \rightarrow x^3 \), and multiply the subsequent identity by \(-a^{-1}x\) to get

\[
(a - a^{-1}) x \prod_{n=1}^{\infty} \frac{(1 - x^{6n}) (1 - a^2 x^{6n}) (1 - a^{-2} x^{6n})}{(1 - ax^{6n-3}) (1 - a^{-1} x^{6n-3})} = \sum_{n=-\infty}^{\infty} (-1)^n x^{(3n+1)^2} (a^{3n+1} - a^{-3n-1}).
\]
Let $F(a, x)$ denote the left side of the foregoing identity, and for a complex variable $z$, regard $zD_z$ as an operator, where $D_z$ denotes derivation with respect to $z$. Then,

$$
(aDa)^2 \{ F(a, x) \} = \sum_{n=0}^{\infty} (-1)^n (3n+1)^2 (a^{3n+1} - a^{-3n-1})
$$

We now use the technique of logarithmic differentiation to evaluate the leftmost and rightmost members of (2.2), cancel $F(a, x)$ in the resulting identity, and then let $x \to x^{1/3}$ to get

$$
\left\{ \frac{a + a^{-1}}{a - a^{-1}} - 2 \sum_{k=1}^{\infty} \frac{x^{2k}}{1 - x^{2k}} (a^{2k} - a^{-2k}) + \sum_{k=1}^{\infty} \frac{x^k}{1 - x^{2k}} (a^k - a^{-k}) \right\}^2
$$

$$
= 1 + \frac{4}{(a - a^{-1})^2} + 4 \sum_{k=1}^{\infty} \frac{kx^{2k}}{1 - x^{2k}} (a^{2k} + a^{-2k}) - \sum_{k=1}^{\infty} \frac{kx^k}{1 - x^{2k}} (a^k + a^{-k})
$$

$$
- 6 \sum_{k=1}^{\infty} \frac{kx^{2k}}{1 - x^{2k}} + 6 \sum_{k=1}^{\infty} \frac{x^{2k}}{1 - x^{2k}} (a^{2k} + a^{-2k}) + 3 \sum_{k=1}^{\infty} \frac{x^k(1 + x^{2k})}{1 - x^{2k}} (a^k + a^{-k}).
$$

In the foregoing identity let $a = e^{i\theta}/2$, subject to the stated restrictions. We simplify the resulting identity, and finally divide by $-16$ to arrive at identity (1.1).

CONCLUDING REMARKS. The forerunner of all identities of type (1.1) is a celebrated one due to Ramanujan [2, p. 139], viz.,

$$
\left\{ \frac{(1/4) \cot (\theta/2) + \sum_{k=1}^{\infty} \frac{x^k \sin k\theta}{1 - x^k}}{1 - x^k} \right\}^2
$$

$$
= \left\{ \frac{(1/4) \cot (\theta/2)}{1 - x^k} \right\}^2 + \sum_{k=1}^{\infty} \frac{x^k \cos k\theta}{1 - x^k} + \frac{1}{2} \sum_{k=1}^{\infty} \frac{kx^k}{1 - x^k} (1 - \cos k\theta),
$$

with the same restrictions on $\theta$ and $x$. Ramanujan himself made substantial applications of his identity to the theory of elliptic modular functions. However, the most familiar application of the identity is perhaps that of Hardy and Wright [3, pp. 311-314]. These authors use the identity to establish Jacobi’s formula for the number $r_4(n)$ of representations of a natural number $n$ by sums of four squares. Ewell [4] shows that the method of this note permits an easy and straightforward derivation of Ramanujan’s identity. Moreover, the method also reveals two additional trigonometrical identities of this type.

REFERENCES
