ON RADII OF CONVEXITY AND STARLIKENESS OF SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT. Let $P[A, B]$, $-1 \leq B < A \leq 1$, be the class of functions p such that $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$. Let $P(\alpha_1)$ be the class of functions with positive real part greater than α_1 , $0 \leq \alpha_1 \leq 1$. It is clear that $P[A, B] \subset P\left(\frac{1-A}{1-B}\right) \subset P[1, -1]$. The principal results in this paper are the determination of the radius of β -starlikeness and β -convexity of $f(z)$ with $\beta = \frac{1-A}{1-B}$, when $f(z)$ is restricted to certain classes of univalent and analytic functions related with $P[A, B]$.

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1. INTRODUCTION.

Let f be analytic in $E = \{z : |z| < 1\}$, and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

A function g , analytic in E , is called subordinate to a function G if there exists a Schwarz function $w(z)$, $w(z)$ analytic in E with $w(0) = 0$ and $|w(z)| < 1$ in E , such that $g(z) = G(w(z))$.

In [1], Janowski introduced the class $P[A, b]$. For A and B , $-1 \leq B < A \leq 1$, a function p , analytic in E with $p(0) = 1$ belongs to the class $P[A, B]$ if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$.

Also $C[A, B]$ and $S^*[A, B]$ denote the classes of functions, analytic in E and given by (1.1) such that $\frac{zf'(z)}{f(z)} \in P[A, B]$ and $\frac{zf'(z)}{f(z)} \in P[A, B]$ respectively. For $A = 1$, and $B = -1$, we note that $C[1, -1] = C$ and $S^*[1, -1] = S^*$, the classes of convex and starlike functions in E . Also $S^*[A, B] \subset S^*\left(\frac{1-A}{1-B}\right) \subset S^*[1, -1]$ and $C[A, B] \subset C\left(\frac{1-A}{1-B}\right) \subset C[1, -1]$, where $S^*\left(\frac{1-A}{1-B}\right)$ and $C\left(\frac{1-A}{1-B}\right)$ denote the classes of starlike and convex functions of order $\frac{1-A}{1-B}$ respectively. These classes were first introduced by Robertson in [2].

A function f , analytic in E and given by (1.1), is said to be in the class $R_k[A, B]$, $-1 \leq B < A \leq 1$, if and only if

Hence

$$\frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} = p(z) + \frac{zp'(z)}{p(z)} - \frac{1-A}{1-B}$$

Using Lemma 2.3 for $\alpha = 1 = \beta$, we have for $R_1 \leq R_2$

$$\begin{aligned} \operatorname{Re} \left[\frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} \right] &\geq \frac{1-(3A-B)r + A^2r^2}{(1-Ar)(1-Br)} - \frac{1-A}{1-B} \\ &= \frac{A-B}{1-B} \left[\frac{1-(2+A-B)r + Ar^2}{(1-Ar)(1-Br)} \right], \end{aligned}$$

and this implies that $\operatorname{Re} \left[\frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} \right] \geq 0$ for $|z| < r_0$, where r_0 is given by (3.1). The inequality $R_1 < R_2$ is satisfied whenever $T(r) = 1 - (2+A-B)r + Ar^2 \geq 0$. But $T(0) = 1 > 0$ and $T(1) = B - 1 < 0$. So $T(r)$ has at least one root in $(0,1)$. Let r_0 , given by (3.1) be that root of $T(r) = 0$. Then in $[0, r_0]$, $R_1 < R_2$ and hence $f \in C\left(\frac{1-A}{1-B}\right)$ for all z with $|z| = r \leq r_0 < 1$.

This result is sharp for the function $f_0 \in S^*[A, B]$ such that

$$\frac{zf'_0(z)}{f_0(z)} = \frac{1+Az}{1+Bz}$$

THEOREM 3.2. Let $g \in S^*[A, B]$ and let $\frac{g'(z)}{g(z)} \in P[A, B]$. Then $f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where r_0 is given by (3.1).

PROOF. $zf'(z) = g(z)p(z)$, $p \in P[A, B]$.

This gives us

$$\frac{(zf'(z))'}{f'(z)} = \frac{zg'(z)}{g(z)} + \frac{zp'(z)}{p(z)}$$

Applying the usual inequalities, we obtain

$$\begin{aligned} \operatorname{Re} \left[\frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} \right] &\geq \frac{1-Ar}{1-Br} - \frac{(A-B)r}{(1-Ar)(1-Br)} - \frac{1-A}{1-B} \\ &= \frac{(A-B)[1-(2+A-B)r + Ar]}{(1-B)(1-Ar)(1-Br)} \end{aligned}$$

Hence we obtain the required result that $f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$ and r_0 is given by (3.1).

THEOREM 3.3. Let $g \in S^*[A, B]$ and $\frac{g'(z)}{g(z)} \in P[A, B]$. Then $\frac{(zf'(z))'}{g'(z)} \in P\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where r_0 is given by (3.1).

PROOF. We have $zf'(z) = g(z)p(z)$, $p \in P[A, B]$ and so

$$\frac{(zf'(z))'}{g'(z)} = p(z) + \frac{g(z)}{zg'(z)} \cdot zp'(z)$$

Thus

$$\begin{aligned} \operatorname{Re} \left[\frac{(zf'(z))'}{g'(z)} - \frac{1-A}{1-B} \right] &\geq \operatorname{Re} p(z) \left[1 - \frac{(1-Br)}{(1-Ar)} \cdot \frac{(A-B)r}{(1-Ar)(1-Br)} \right] - \frac{1-A}{1-B} \\ &\geq \frac{(1-Ar)}{(1-Br)} \left[\frac{1-(3A-B)r + A^2r^2}{(1-Ar)(1-Br)} \right] - \frac{1-A}{1-B} \\ &= \frac{(A-B)}{(1-B)} \left[\frac{1-(2+A-B)r + Ar^2}{(1-Ar)(1-Br)} \right] \end{aligned}$$

$$f(z) = \frac{(S_1(z))^{\frac{k}{4} + \frac{1}{2}}}{(S_2(z))^{\frac{k}{4} - \frac{1}{2}}}, \quad S_1, S_2 \in S^*[A, B]. \quad (1.2)$$

Clearly $k \geq 2$ and $R_k[A, B] = S^*[A, B]$. Also $R_k[1, -1] = U_k$, the class of functions with bounded radius rotation discussed in [3].

Similarly we can define the class $V_k[A, B]$ as follows. A function f , analytic in E and given by (1.1) belongs to $V_k[A, B]$, $k \geq 2$, if and only if

$$f'(z) = \frac{(S_1(z)/z)^{\frac{k}{4} + \frac{1}{2}}}{(S_2(z)/z)^{\frac{k}{4} - \frac{1}{2}}}, \quad S_1, S_2 \in S^*[A, B] \quad (1.3)$$

From (1.2) and (1.3), it is clear that

$$f \in V_k[A, B] \quad \text{if and only if} \quad zf' \in R_k[A, B] \quad (1.4)$$

It may be noted that $V_2[A, B] = C[A, B]$ and $V_k[1, -1] = V_k$, the class of functions of bounded rotation first discussed by Paatero [4].

2. PRELIMINARY RESULTS

LEMMA 2.1 [5] Let $p \in P[A, B]$. Then

$$\frac{1 - Ar}{1 - Br} \leq \operatorname{Re} p(z) \leq |p(z)| \leq \frac{1 + Ar}{1 + Br}$$

The following is the extension of Libera's result [6].

LEMMA 2.2. Let N and D be analytic in E , D map onto a many-sheeted starlike region. $N(0) = 0 = D(0)$ and $\frac{N(z)}{D(z)} \in P[A, B]$. Then $\frac{N(z)}{D(z)} \in P[A, B]$. For the proof of this result we refer to [5].

LEMMA 2.3. [7] Let $p \in P[A, B]$. Then, for $z \in E$, $\alpha \geq 0$ and $\beta \geq 0$, we have

$$\operatorname{Re} \left\{ \alpha p(z) + \beta \frac{zp'(z)}{p(z)} \right\} \geq \begin{cases} \frac{\alpha - \{\beta(A - B) + 2\alpha A\}r + \alpha A^2 r^2}{(1 - Ar)(1 - Br)}, & R_1 \leq R_2 \\ \beta \frac{A + B}{A - B} + \frac{2[(L_1 K_1)^{1/2} - \beta(1 - ABR^2)]}{(A - B)(1 - r^2)}, & R_2 \leq R_1 \end{cases}$$

where

$$R_1 = \left(\frac{L_1}{K_1} \right)^{1/2}, \quad R_2 = \frac{1 - Ar}{1 - Br}, \quad L_1 = \beta(1 - A)(1 + Ar^2)$$

and

$$K_1 = \alpha(A - B)(1 - r^2) + \beta(1 - B)(1 + Br^2).$$

This result is sharp.

3. MAIN RESULTS.

THEOREM 3.1. Let $f \in S^*[A, B]$. Then $f \in C\left(\frac{1-A}{1-B}\right)$ for

$$|z| < r_0 = \frac{2}{(2 + A - B) + \sqrt{(2 + A - B)^2 - 4A}} \quad (3.1)$$

This result is sharp.

PROOF. We have $zf'(z) = f(z)p(z)$, $p \in P[A, B]$

Hence $\frac{(zf'(z))'}{f'(z)} \in P\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where r_0 is given by (3.1).

Our next result is about the radius of convexity problem for the class $V_k[A, B]$.

THEOREM 3.4. Let $f \in V_k[A, B]$, $k \geq 2$. Then $f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$, where

$$r_1 = \frac{4}{k(1-B) + \sqrt{k^2(1-B)^2 + 16B}} \quad (3.2)$$

PROOF. Since $f \in V_k[A, B]$, we have from (1.3)

$$f'(z) = \frac{(S_1(z)/z)^{\frac{k}{4} + \frac{1}{2}}}{(S_2(z)/z)^{\frac{k}{4} - \frac{1}{2}}}, \quad S_1, S_2 \in S^*[A, B]$$

This implies that

$$\frac{(zf'(z))'}{f'(z)} = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \quad p_1, p_2 \in P[A, B]$$

so

$$\begin{aligned} \operatorname{Re} \left[\frac{(zf'(z))'}{f'(z)} - \frac{1-A}{1-B} \right] &\geq \left(\frac{k}{4} + \frac{1}{2}\right) \left(\frac{1-Ar}{1-Br} \right) - \left(\frac{k}{4} - \frac{1}{2}\right) \left(\frac{1+Ar}{1+Br} \right) - \frac{1-A}{1-B} \\ &= \frac{(A-B) - \frac{k}{2}(1-B)(A-B)r - B(A-B)r^2}{(1-B)(1-B^2r^2)} \end{aligned}$$

Hence $f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$, r_1 is given by (3.2).

From Theorem 3.4 and relation (1.4) we have the following:

THEOREM 3.5. Let $f \in R_k[A, B]$. Then $f \in S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ where r_1 is given by (3.2).

THEOREM 3.6. Let α and m be any positive integers and $f \in R_k[A, B]$. Then the function F defined by

$$(F(z))^\alpha = \frac{\alpha+m}{z^m} \int_0^z t^{m-1} (f(t))^\alpha dt \quad (3.3)$$

belongs to $S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$, r_1 is given by (3.2).

PROOF. Let $J(z) = \int_0^z t^{m-1} (F(t))^\alpha dt$ and so

$$(F(z))^\alpha = \frac{\alpha+m}{z^m} J(z),$$

and

$$\frac{\alpha z F'(z)}{F(z)} = \frac{z J'(z)}{J(z)} - m$$

or

$$\begin{aligned} \frac{z F'(z)}{F(z)} &= \frac{1}{\alpha} \frac{z J'(z) - m J(z)}{J(z)} = \frac{N(z)}{D(z)} \\ N(0) &= 0 = D(0) \end{aligned}$$

By a result of Bernardi [8] and Theorem 3.5, $D(z)$ is a $(m + \alpha - 1)$ -valent starlike function for $|z| < r_1$. Also

$$\frac{N'(z)}{D'(z)} = \frac{1}{\alpha} \left\{ \frac{(zJ'(z))' - mJ'(z)}{J'(z)} \right\} \\ = \frac{zf'(z)}{f(z)}$$

Now, by Theorem 3.5, $f \in S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ and this implies that $\frac{N'(z)}{D'(z)} \in P\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$. Hence

$$\frac{N(z)}{D(z)} \in P\left(\frac{1-A}{1-B}\right) \quad \text{for } |z| < r_1, \quad \text{see [8].}$$

This proves our result.

Similarly, we can prove the following:

THEOREM 3.7. Let α and m be positive integers and $f \in V_k[A, B]$. Let F be defined by (3.3). Then $f \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ where r_1 is given by (3.2).

We now prove:

THEOREM 3.8. Let f and $g \in R_k[A, B]$ and, for α, m positive integers, let F be defined as

$$(F(z))^\alpha = \frac{(m + \alpha)}{(g(z))^m} \int_0^z t^{m-1} (f(t))^\alpha dt \quad (3.4)$$

Then $F \in S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$

where $r_0 = \min(r_1, r_2)$, r_1 is given by (3.2) and r_2 is the least positive root of the equation

$$\{(1-B) - \alpha(1-A) - \{(A-B)(1+2m)\}r + \{(A-B) + 2m(A-B) + \alpha(1-A)\}r^2\} = 0, \quad (3.5)$$

PROOF. Let $J_1(z) = \frac{\alpha+m}{z^m} \int_0^z t^{m-1} (f(t))^\alpha dt$.

Then $(F(z))^\alpha = \left(\frac{z}{g(z)}\right)^m J_1(z)$, where by Theorem 3.6, $J_1 \in S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$.

So

$$\frac{\alpha z F'(z)}{F(z)} = \frac{z J_1'(z)}{J_1(z)} + m \left(1 - \frac{z g'(z)}{g(z)}\right)$$

Thus

$$Re \left[\frac{z F'(z)}{F(z)} - \frac{1-A}{1-B} \right] \geq \frac{1}{\alpha} \left[\left(1 + \frac{B-A}{1-B} r\right) / (1+r) \right] + \left[\left(\frac{2m}{\alpha} (B-A) r \right) / (1-R) \right] - \frac{1-A}{1-B} \\ - \left[\frac{\{(1-B - \alpha + \alpha A) + [(B-A)(1+2m)]r + [(A-B) + 2m(A-B) + \alpha(1-A)]r^2\}}{\alpha(1-B)(1-r^2)} \right]$$

This implies $Re \frac{z F'(z)}{F(z)} \geq \frac{1-A}{1-B}$ for $|z| < r_2$, where r_2 is the least positive root of (3.5). Hence $F \in S^*\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where $r_0 = \min(r_1, r_2)$.

Similarly, we have the following:

THEOREM 3.9. Let f and $g \in V_k[A, B]$ and, for α, m positive integers, let F be defined by (3.4).

Then $F \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_0$, where r_0 is as given in Theorem 3.8.

THEOREM 3.10. Let $g \in V_k[A, B]$ and $\frac{f'(z)}{g'(z)} \in P[A, B]$ and let F be defined by

$$F(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} f(t) dt,$$

where m is any positive integer. Then there exists a function G such that

$$\frac{F'(z)}{G'(z)} \in P\left(\frac{1-A}{1-B}\right), \quad G \in C\left(\frac{1-A}{1-B}\right)$$

for $|z| < r_1$, where r_1 is given by (3.2).

PROOF. Let

$$G(z) = \frac{m+1}{z^m} \int_0^z t^{m-1} g(t) dt.$$

Then, by Theorem 3.7 with $\alpha = 1$, $G \in C\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ and r_1 is defined by (3.2). Now

$$\begin{aligned} \frac{F'(z)}{G'(z)} &= \frac{z^m f(z) - m \left(\int_0^z t^{m-1} f(t) dt \right)}{z^m g(z) - m \left(\int_0^z t^{m-1} g(t) dt \right)} \\ &= \frac{\int_0^z t^m f'(t) dt}{\int_0^z t^m g'(t) dt} = \frac{N(z)}{D(z)} \end{aligned}$$

Also

$$\frac{N'(z)}{D'(z)} = \frac{f'(z)}{g'(z)} \in P[A, B] \quad \text{for } |z| < r_1$$

Thus, by Lemma 2.2, we have $\frac{N(z)}{D(z)} \in P[A, B] \subset P\left(\frac{1-A}{1-B}\right)$ for $|z| < r_1$ and this proves our result.

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