A NON-UNIQUENESS THEOREM IN THE THEORY OF VORONOI SETS

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ABSTRACT. It is shown that two distinct, bounded, open subsets of $\mathbb{R}$ may possess the same Voronoi set.

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1. INTRODUCTION

Let $\{D_i\}_{i=1}^{n}$ be a finite collection of non-empty, bounded, open and simply connected subsets of $\mathbb{R}^2$ which satisfy $D_i \subset D_0$, $D_i \neq D_0$, $1 \leq i \leq n$ and $D_i \cap D_j = \emptyset$, $1 \leq i < j \leq n$. Then if we define $\Omega = D_0 \setminus \bigcup_{i=1}^{n} D_i$, $\Omega$ is a non-empty, bounded, open and connected subset of $\mathbb{R}^2$ with boundary $\partial \Omega = \bigcup_{i=0}^{n} \partial D_i$ (loosely speaking, $\Omega$ is a domain $D_0$ containing "obstacles" $D_i$, $1 \leq i \leq n$.) The the following definition of the Voronoi diagram $\text{Vor}(\Omega)$ of $\Omega$ is taken from [1].

For any $(x,y) \in \Omega$, define $\text{Near}(x,y)$ as the set of points in $\partial \Omega$ closest to $(x,y)$. ("Closest to" is, of course, defined in terms of ordinary Euclidean distance in the plane.) Since $\partial \Omega$ is closed, $\text{Near}(x,y)$ is always non-empty.

The Voronoi diagram $\text{Vor}(\Omega)$ of $\Omega$ is then defined to be the set of points

$$\{(x,y) \in \Omega : \text{Near}(x,y) \text{ contains more than one point}\}.$$  

$\text{Vor}(\Omega)$ is used in [1] in connection with motion planning problems.

Clearly given the sets $\{D_i\}$, $\text{Vor}(\Omega)$ is unique. However, here we take the opposite point of view and consider the construction of the sets $\{D_i\}$ from a given Voronoi diagram.

A preliminary question that one might ask is: could it be possible for two collections $\{D_i\}$ and $\{D_i^\prime\}$ to have the same Voronoi diagrams? It is easy to see that the answer is yes: for $0 < \epsilon < 1$ let

$$D_0^\epsilon = \{(x,y) \mid x^2+y^2 < (1+\epsilon)^2\} \quad \text{and} \quad D_1^\epsilon = \{(x,y) \mid x^2+y^2 < (1-\epsilon)^2\}.$$
Then if \( \Omega^c = D_0 \setminus \overline{D}_1 \), Vor(\( \Omega^c \)) is the unit circle, centre the origin, whatever the value of \( \epsilon \) might be.

A more subtle question is the following: Suppose \( D_0' = D_0' \), then is it possible for two different collections \( \{D_i\} \) and \( \{D_i'\} \) to have the same Voronoi diagram? In other words, what we are asking is whether, given a fixed domain \( D_0 \), it is possible to arrange two different sets of obstacles within \( D_0 \), both of which produce the same Voronoi diagram. We show the answer is again in the affirmative.

2. THE EXAMPLE

Let
\[
D_0 = \{(x,y) \mid |x| < 4, |y| < 4\}
\]
\[
D_1 = \{(x,y) \mid |x| < 3, 1 < y < 3\}
\]
\[
D_2 = \{(x,y) \mid |x| < 3, -3 < y < -1\}.
\]

Then \( \Omega \) and Vor(\( \Omega \)) (where \( \Omega = D_0 \setminus \overline{D}_1 \cup \overline{D}_2 \)) are depicted in Figure 1. Note in particular that Vor(\( \Omega \)) contains the line segment \( \{(x,0) \mid |x| \leq 1\} \).

![Figure 1 - Vor(\( \Omega \)) is denoted by the dashed line](image-url)
We modify $D_1$ and $D_2$ as follows. Let $C = \{(x,y) \mid x^2 + y^2 \leq 2\}$ and put $D'_1 = D_1 \setminus C$, $D'_2 = D_2 \setminus C$. Then if $\Omega' = D_0 \setminus D'_1 \cup D'_2$, $\text{Vor}(\Omega) = \text{Vor}(\Omega')$, (see Figure 2).

![Figure 2 - Vor(\Omega') is denoted by the ashed line](image)

To see that the Voronoi diagrams of $\Omega$ and $\Omega'$ are indeed the same first note that it suffices to consider those points $(x,y)$ in $\Omega'$ for which $|x| \leq 1$ and $|y| \leq \sqrt{2}$ since for any other $(x,y) \in \Omega'$, $\text{Near}(x,y)$ will be unchanged by the modifications made to $D_1$ and $D_2$. To begin with, consider those points within the triangle whose vertices are $(-1,0)$, $(0,0)$ and $(-1,1)$. It is clear that if $(x,y)$ is such a point then $\text{Near}(x,y) = \{(-1,1)\}$ and so $(x,y) \notin \Omega'$. The same conclusion is true for the points in $\Omega'$ which lie on the straight lines joining $(-1,1)$ to $(-1,0)$ and $(-1,1)$ to $(0,0)$, (excluding the endpoints of those lines). Next consider the points $(x,0)$ where $-1 \leq x < 0$. For such a point $\text{Near}(x,0) = \{(-1,1), (-1,-1)\}$ and so $(x,0) \notin \text{Vor}(\Omega')$. It is also clear that $(0,0) \in \text{Vor}(\Omega')$. Now consider those points within the sector of $C$ which has vertices $(0,0)$, $(-1,1)$ and $(0,\sqrt{2})$. If $(x,y)$ is such a point then it is easy to see that $\text{Near}(x,y)$ consists of the single point obtained by projecting the straight line joining $(0,0)$ to $(x,y)$ until it intersects $D'_1$. The same conclusion is true for the points on the straight line between $(0,0)$ and $(0,\sqrt{2})$ (excluding the endpoints of course). The results for
the remaining points in $\Omega'$ follow immediately from the symmetry of $\Omega'$. Hence $\text{Vor}(\Omega) = \text{Vor}(\Omega')$.

A possible weakness of this example is that the sets $D_1'$ and $D_2'$ are not convex. The answer to the same question as that posed in 5.1 but with the additional hypothesis that all the sets in $\{D_1\}$ and $\{D_1'\}$ be convex would appear to be unknown.

REFERENCES
