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BANACH-MACKEY SPACES

JING HUI QIU

Department of Mathematics, University of Suzhou, Suzhou, Jiang Su Province, China

KELLY McKENNON

Department of Mathematics, Washington State University Pullman, Washington, USA

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ABSTRACT. In recent publications the concepts of fast completeness and local barreledness have been shown to be related to the property of all weak-* bounded subsets of the dual (of a locally convex space) being strongly bounded. In this paper we clarify those relationships, as well as giving several different characterizations of this property.

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1. INTRODUCTION.

In [1], it is claimed that in a locally convex space E all $\sigma(E',E)$-bounded sets are $\beta(E',E)$-bounded if and only if E is fast complete. In [2] Kučera and Gilsdorf pointed out that the "only if" part is not correct and proposed a notion "locally barreled" which is weaker than fast completeness. They proved that if E is locally barreled, then all $\sigma(E',E)$-bounded sets are $\beta(E',E)$-bounded. They also formulated a certain property (P) and showed that E is locally barreled if it satisfies property (P) and if all $\sigma(E',E)$-bounded sets are $\beta(E',E)$-bounded. Thus, when E has property (P), a necessary and sufficient condition for all $\sigma(E',E)$-bounded sets to be $\beta(E',E)$-bounded is for E to be locally barreled. In [3] Gilsdorf proved that whenever E is locally barreled, then the families of weakly and strongly bounded subsets of $\mathfrak{L}(E,F)$ are identical and that, when property (P) holds, the two statements are equivalent.

In the present paper we give a number of necessary and sufficient conditions, as well as some sufficient conditions, for weak-* bounded subsets to be strongly bounded, and then investigate the relationships between them. In particular, we prove that whenever all $\sigma(E',E)$-bounded sets are $\beta(E',E)$-bounded and F is any locally convex space, then the families of weakly and strongly bounded subsets of $\mathfrak{L}(E,F)$ are

identical (which yields the above-cited result of [3] as a direct consequence).

Each Fréchet space is locally barreled, but we present below an example of a barreled space which is not locally barreled. This shows that all weak-* bounded sets may be strongly bounded without a space being locally barreled.

It turns out that property (P) is rather demanding. In particular we show that if all weak-* bounded sets are strongly bounded, that property (P) does not hold <u>unless each linear functional is continuous</u> (which is never the case, for instance, in an infinite dimensional Fréchet space). Thus there are even many Banach spaces (which are locally barreled and fast complete) which do not have property (P)

2. BANACH-MACKEY SPACES (VARIOUS DESCRIPTIONS).

In [1]-[3] some conditions for weak-* bounded sets to be strongly bounded have been investigated. For brevity, we denote a Hausdorff locally convex linear topological space by the abbreviation l.c.s. As in [4], Def. 10-4-3, we call a l.c.s. $(E,\mathcal{T})$ a Banach-Mackey space if all $\sigma(E,E')$-bounded subsets are $\beta(E,E')$-bounded. We begin by giving a number of necessary and sufficient conditions for weak-* bounded sets of be strongly bounded.

THEOREM 1. Let $(E,\mathcal{T})$ be a l.c.s. The following statements are pairwise equivalent:

(S1) all $\sigma(E',E)$-bounded subsets of E' are $\beta(E',E)$-bounded;

(S2) $(E,\mathcal{T})$ is a Banach-Mackey space;

(S3) each barrel in $(E,\mathcal{T})$ is a bornivore (absorbs bounded sets) in $(E,\mathcal{T})$;

(S4) $\beta(E'',E')|_E = \beta(E,E')$;

(S5) for any absolutely convex, bounded, closed subset B of E, the topology on the linear hull E_B of B generated by the Minkowski functional ρ_B of B is finer than that topology $\beta(E,E')$ restricted to E_B;

(S6) for any l.c.s. F and any family S of bounded subsets of E covering E, a subset $\mathfrak{B}$ of the space $\mathfrak{L}(E,F)$ (of continuous linear operators from E to F) is pointwise bounded if and only if it is bounded on each element of S (S-bounded).

PROOF. That (S2) is equivalent to (S1), (S3), and (S4) is proved in [4], Th. 10-4-5, Th. 10-4-7, and Th.10-4-101.

Next we show that (S3) is equivalent to (S5). Let B be as in (S5) and let W be a barrel in $(E,\mathcal{T})$ (so that $E_B\cap W$ is a typical neighborhood of 0 for the topology $\beta(E,E')$ relativized to E_B). To know that W will always absorb B is to know that some positive scalar-multiple of the set $\{x\in E: \rho_B(x)\leq 1\}$ is contained in W, which is to know that (S5) holds.

That (S6) implies (S2) is trivial. We shall complete the proof by assuming that (S3) holds and demonstrating that (S6) follows. Let then F and S be as in (S6), let A be an element of S, let $\mathfrak{B}$ be a pointwise bounded subset of $\mathfrak{L}(E,F)$, and let ρ be a continuous seminorm on F. The set $B\equiv\{x\in E: \rho(T(x))\leq 1\ (\forall T\in\mathfrak{B})\}$ is a barrel in E and so, by (S3), B absorbs bounded subsets of E — in particular B absorbs A. It follows that $\mathfrak{B}$ is bounded on A, which establishes (S6). Q.E.D.

3. BANACH-MACKEY SPACES (SUFFICIENT CONDITIONS).

In the present section we give some sufficient conditions for weak-* bounded sets to be strongly bounded.

DEFINITION 1 (cf. [2] or [3]). Let $B\subseteq E$ be a disk, E_B the linear hull of B, and ρ_B the Minkowski functional of B:

(a) if (E_B,ρ_B) is a Banach space, then B is called a <u>Banach disk</u> and E is said to be <u>fast complete</u> if each bounded subset of E is contained in a bounded Banach disk;

(b) if (E_B,ρ_B) is a barreled normed space, then B is called a <u>barreled disk</u> and E is said to be <u>locally barreled</u> if each bounded subset of E is contained in a closed bounded barreled disk.

THEOREM 2. Let $(E,\mathcal{T})$ be a l.c.s. Each of the following statements implies that E is a Banach-Mackey space:

(1) $(E,\mathcal{T})$ is locally barreled;

(2) $(E,\mathcal{T})$ is semi-reflexive;

(3) for each absolutely summable sequence c_n of scalars and each null sequence x_n in $(E,\mathcal{T})$, the series $\sum_{n=1}^{\infty} c_n x_n$ is convergent.

PROOF. We first show that (1) implies (S3) of Theorem 1. Let W be any barrel and B any bounded subset of $(E,\mathcal{T})$. Then $W\cap E_B$ is is a barrel in E_B (since E_B is continuously embedded in E), a neighborhood of 0 in E_B (by (1)), and so absorbs the bounded subset B of E_B.

That (2) implies (S4) of Theorem 1 is trivial.

We conclude by showing that (3) implies (S3). Suppose that (3) holds and assume that there exists a barrel W in $(E,\mathcal{T})$ which is not a bornivore. Then there is a bounded sequence $\{y_n\}$ such that $y_n\notin nW$ for each $n\in\mathbb{N}$ and so $\{x_n\equiv y_n/n\}$ is a null sequence in the complement of W. Define $T|\ell_1 \to E$ by

$$T(c) \equiv \sum_{n=1}^{\infty} c_n x_n \tag{3.1}$$

for each $c\in\ell_1$. If $f\in E'$, then $M\equiv\sup\{|f(y_n)|:n\in\mathbb{N}\}$ is finite and for all $c,d\in\ell_1$ we have

$$|f(T(c)) - f(T(d))| \le \sum_{n=1}^{\infty}|(c_n-d_n)f(y_n/n)| \le M\cdot\sum_{n=1}^{\infty}|(c_n-d_n)/n| \ . \tag{3.2}$$

It follows that T is $\sigma(\ell_1,c_0)$-$\sigma(E,E')$ continuous on bounded subsets of ℓ_1. Since the closed unit ball D of ℓ_1 is $\sigma(\ell_1,c_0)$-compact, it follows that the image T(D) is $\sigma(E,E')$-compact and so $\beta(E,E')$-bounded as well. But W (being a barrel) then absorbs T(D) which contains the range of the sequence x as a subset: absurd! Thus, (S3) holds. Q.E.D.

DEFINITION 2 (cf. [4], Def. 9-2-8). A l.c.s. E is said to have the <u>convex compactness</u> property if the closed absolutely convex hull of each compact set is compact.

THEOREM 3. If $(E,\mathcal{T})$ has the convex compactness property, then condition (3) of Theorem 2 is satisfied.

PROOF. Assume that $(E,\mathcal{T})$ has the convex compactness property and let x_n and c_n be as in (3) of Theorem 2. Then the range A of x is compact and so the closed absolute convex hull [A] is compact. Let d_n be the sequence c_n divided by its ℓ_1-norm. Then each of the partial sums $\sum_{n=1}^{m} d_n x_n$ is in [A] and so there exists a limit point s of the partial sums. For two finite sums $\sum_{n=1}^{m} d_n x_n$ and $\sum_{n=1}^{m+k} d_n x_n$ and any continuous semi-norm ρ on E we have

$$\rho\Big(\sum_{n=1}^{m+k} d_n x_n - \sum_{n=1}^{m} d_n x_n\Big) \le \sum_{n=m+1}^{m+k}|d_n|\cdot\rho(x_n). \tag{3.3}$$

3.3)

Since $\{\rho(x_n):n\in\mathbb{N}\}$ is bounded, it follows that the sequence of partial sums is Cauchy. Hence $\sum_{n=1}^{\infty} d_n x_n$ exists (and equals s). Q.E.D.

COROLLARY. If a l.c.s. $(E,\mathcal{T})$ is either fast complete or has the convex compactness property, then it is a Banach-Mackey space.

PROOF. If $(E,\mathcal{T})$ is fast complete, it is evidently locally barreled. Thus the

corollary follows from Theorems 1 and 2. Q.E.D.

4. COUNTER-EXAMPLES.

In this section we show that none of the conditions of Theorem 2 is necessary for E to be a Banach-Mackey space.

EXAMPLE 1 (a barreled Banach-Mackey space which is not locally barreled). It is shown in [8] that there exists a Hausdorff inductive limit $(E,\mathcal{T})$ of Fréchet spaces $(E_n,\mathcal{T}_n)$ such that each bounded subset of E is contained in some E_n but that there exists a closed, absolutely convex, bounded subset B of E which is not bounded in any space E_n. Obviously $(E,\mathcal{T})$ is barreled and so is a Banach-Mackey space as well.

Assume that $(E,\mathcal{T})$ were locally barreled. Then we may also assume that the set B above is such that E_B is a barreled space. Denote by ρ_B the Minkowski functional on E_B and let $m\in\mathbb{N}$ be such that $E_B\subseteq E_m$. Let $\{U_n\}$ be a nested neighborhood base of 0 in E_m consisting of closed, absolutely convex sets. Then $\{\frac{B}{n}\cap U_n : n\in\mathbb{N}\}$ generates a metrizable locally convex topology $\mathcal{T}_o$ on E_B which is finer than the norm topology induced by ρ_B. It follows that $(E_B,\mathcal{T}_o)$ is complete ([9] I.1.6). It now follows from the generalized closed graph theorem ([4] Th. 12-5-7) that $\mathcal{T}_o$ is exactly the norm topology induced by ρ_B. Hence each set $\frac{B}{n}\cap U_n$ contains a positive multiple of B, whence follows that each U_n absorbs B. This means that B is bounded in E_m: absurd.

We note that the fact that $(E,\mathcal{T})$ is not locally barreled can also be deduced from [4] T. 12-5-10.

EXAMPLE 2 (a Banach-Mackey space which is not semi-reflexive). Choose any Banach space which is not semi-reflexive.

EXAMPLE 3 (a Banach-Mackey space which does not have property (3) of Theorem 2). It is shown in [5] (cf. also [6] 31.6) that there exists a Hausdorff inductive limit $(E,\mathcal{T})$ of Fréchet spaces $(E_n,\mathcal{T}_n)$ containing a bounded sequence $\{y_n\}$ not contained in any of the spaces E_n.

Assume that (3) of Theorem 2 holds. As in the last paragraph of the proof of Theorem 2, we have that T(D) is $\sigma(E,E')$-compact and so a Banach disk in $(E,\mathcal{T})$. By the localization theorem for strictly webbed spaces ([7] 35.6), T(D) is contained in E_m for some $m\in\mathbb{N}$: absurd!

5. BANACH-MACKEY SPACES AND PROPERTY P.

In [2], Kučera and Gilsdorf formulated a property (P) and proved that if E has property (P) and if each $\sigma(E',E)$-bounded set is $\beta(E',E)$-bounded, then E is locally barreled. Thus if a space has property (P), it is a Banach-Mackey space if and only if it is locally barreled. We prove below that property (P) is actually quite demanding. For reference we set down this property:

(P) for each absolutely convex, bounded, closed subset B of E, there exists a barrel W in E such that $B=W\cap E_B$.

The following result seemed rather surprising.

THEOREM 4. Let $(E,\mathcal{T})$ be a Banach-Mackey space with property (P). Then each linear functional on E is continuous.

PROOF. Let B be any absolutely convex, bounded, closed subset of E and denote by ρ_B the Minkowski functional of B. By Theorem 1 (S6) it follows that the topology $\mathcal{T}_B$ induced by ρ_B is finer than the relativized $\beta(E,E')$-topology. But property (P) implies $\mathcal{T}_B$ is coarser as well—hence $\mathcal{T}_B$ is the relativized topology $\beta(E,E')$.

Assume there is a discontinuous linear functional f on $(E,\mathcal{T})$. Then there exists a

bounded set D contained in no finite dimensional sub space of E ([4] Prob. 4-4-109). Let $\{y_n\}$ be a sequence of linearly independent elements of D and denote y_n/n as x_n for each $n \in \mathbb{N}$. It follows from Theorem 1 (S3) that D is $\beta(E,E')$-bounded and so $\{x_n\}$ is a null sequence in $(E,\beta(E,E'))$. Let B the closed, absolutely convex hull of the range of this null sequence, so that (in particular) B is $\beta(E,E')$-precompact. From the first paragraph of this proof follows that B is precompact in E_B. But a precompact normed space must be finite-dimensional: absurd! Q.E.D.

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