FROM PATHS TO STARS

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ABSTRACT. The number of cycles in the complement $T'$ of a tree $T$ is known to increase with the diameter of the tree. A similar question is raised and settled for the number of complete subgraphs in $T'$ for a special class of trees via Fibonacci numbers. A structural characterization of extremal trees is also presented.

KEY WORDS AND PHRASES. Cycles, Complete subgraphs, Trees.

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1. INTRODUCTION.

Among all trees $T$ of order $n$, the number $c(T')$ of all cycles in the complement $T'$ and the structural characterization of those trees which optimize $c(T')$ have been dealt with in [1,2]. The same problem was solved in [3] for the number $i(T')$ of all complete subgraphs in the complement of an arbitrary $T$. It turns out that among all $n$-trees $T$, the path $P_n (n \geq 9)$ has both the maximum number of cycles [1] and the minimum number of complete subgraphs in its complement [3]. The star $S_n$ also maximizes both $c(T')$, $5 \leq n \leq 8$ [1] and $i(T')$ for $n \geq 4$ [2]. It minimizes $c(T')$ for $n \geq 9$.

The problem of characterizing the $n$-vertex trees $T$ for extremal values with respect to $c(T')$ or $i(T')$ loses some structural significance in the generality of $T$. Suppose we consider a class of trees which keep out all paths and stars, for example, the class $\mathcal{T}_3$ of all those trees $T_3$ having exactly three endvertices. What structural similarities between $c(T_3')$ and $i(T_3')$ are inherited from $c(T')$ and $i(T')$? We recall that the diameter of a graph $G$ is the maximum distance $d(u,v)$ taken over all pairs of vertices $u,v$ in $G$. The following theorem [1, p.93] relates $c(T')$ with the diameter of $T$.

THEOREM 1. For each $n \geq 6$ and every tree $T$ of order $n$ and diameter $d$, $4 \leq d \leq n-2$, there is a tree $T_4$ of order $n$ and at least diameter $d+1$ such that $c(T') < c(T_4')$.

2. TREES WITH THREE ENDVERTICES.

Utilizing enumerative techniques [2] we conclude that among all trees $T_3$ of order $n$, the
A tree with the smallest number of cycles in its complement is a tree with the smallest diameter as shown in Figure 1. Moreover, the tree \( T_3 \)

\[
n = \begin{cases} 
3k - 1 & \\
3k & d = \begin{cases} 
2k - 1 \\
2k 
\end{cases} \\
3k + 1
\end{cases}
\]

Figure 1. A tree with three endvertices and minimum \( c(T'_3) \).

with the largest number of cycles in its complement is a tree with the largest diameter as shown in Figure 2.

Figure 2. A tree with maximum \( c(T'_3) \).

So, among all trees with three endvertices, of order \( n \), the direct relationship between \( c(T'_3) \) and the diameter of \( T_3 \) is inherited from the class of all trees, i.e., \( \min c(T'_3) \) and \( \max c(T'_3) \) are still associated with the smallest and the largest diameters of \( T_3 \), respectively.

Can we make the same claim about \( i(T'_3) \), the number of complete subgraphs in the complement of \( T_3 \)? We recall that when \( T \) is arbitrary, \( i(T') \) is maximum when the diameter of \( T \) is minimum (\( T \) is a star) and \( i(T') \) is minimum when the diameter is maximum (\( T \) is a path).

Does this relationship between \( i(T') \) and the diameter of \( T \) remain true when \( T \) is restricted to \( T_3 \)? To this end, we need the concept of a Fibonacci number \( f(G) \) of a graph \( G \).

According to [4, p. 45], the total number of subsets of \( \{1, 2, 3, \ldots, n\} \) such that no two elements are adjacent is \( F_{n+1} \), where \( F_n \) is the \( n \)th Fibonacci number, which is defined by

\[
F_0 = F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.
\]

The sequence \( \{1, 2, 3, \ldots, n\} \) can be regarded as the vertex set of the path \( P_n \). This definition covers the empty graph also; so, \( f(G) = i(G') \). We note that \( i(P'_0) = 1, i(P'_1) = 2, i(P'_2), \ldots, i(P'_n) = F_{n+1} \).

3. MAIN RESULTS

If \( T_3 \) is a tree with three endvertices, then it has a unique vertex \( u \) of degree three. We count \( i(T'_3) \) [5] by considering two disjoint sets of complete subgraphs of \( T'_3 \), say \( S_1 \) and \( S_2 \), where \( S_1 \) is the set of those complete subgraphs not containing the vertex \( u \), and \( S_2 \) consists of those that do contain \( u \). Let \( v_1, v_2 \) and \( v_3 \) be the three vertices adjacent to \( u \) in \( T_3 \). We have

\[
i(T'_3) = |S_1| + |S_2| = i(T_3 - u)' + i(T_3 - v_1 - v_2 - v_3)'.
\]

If \( n = 3k + 1, T_3 - u \) is a union of three disjoint paths on \( 3k \) vertices, where \( T_3 - v_1 - v_2 - v_3 \)
is also a union of three disjoint paths on $3k - 3$ vertices together with the isolated vertex $u$ (see Figure 1). The following theorem on Fibonacci numbers shows that $i(T')$ is minimized and maximized by the trees in Figures 3a and 3b, respectively. This shows

![Extremal trees in $F_3$.](image)

that the inverse relationship between $i(T')$ and the diameter of $T$ is not inherited in the class of trees $F_3$ with exactly three endvertices.

**THEOREM 2.** Let $n$ be an integer $\geq 7$. Then among all summands $r_1, s_1, t_1$ and $r_2, s_2, t_2$ satisfying (i) $r_1 + s_1 + t_1 = n + 2$, (ii) $r_1 = r_2 + 1, s_1 = s_2 + 1, t_1 = t_2 + 1$, and $r_1, s_1, t_1 \geq 2$ we have

the sum of the two products $F_{r_1} F_{s_1} F_{t_1} + F_{r_2} F_{s_2} F_{t_2}$ of the Fibonacci numbers $F_{r_1}, F_{s_1}, F_{t_1}$ and $F_{r_2}, F_{s_2}, F_{t_2}$ is

(a) minimum if $r_1 = s_1 = 3$ and $t_1 = n - 4$ and

(b) maximum if $r_1 = s_1 = 2$ and $t_1 = n - 2$.

**PROOF.** The order of growth of $F_n$ is governed by the golden ratio $\tau = (1 + \sqrt{5})/2$. Moreover, $F_n \approx cr^n$ where $c = \tau/\sqrt{5}$ and $F_{n+1} \approx \tau F_n$. We have $r_1 + s_1 + t_1 = n+2$ and $r_2 + s_2 + t_2 = n-1$, and for large $n$,

$$F_{r_1} F_{s_1} F_{t_1} + F_{r_2} F_{s_2} F_{t_2} \approx (\tau F_{r_1})(\tau F_{s_1})(\tau F_{t_1}) + F_{r_2} F_{s_2} F_{t_2} = (1 + \tau^3) F_{r_2} F_{s_2} F_{t_2}$$

$$\approx \left(5.236068\cdots\right)(c^2 \tau^{n+1+2}) \approx 13.59764677\cdots \tau^{n-5} > 13.43181071\cdots \tau^{n-5}$$

$$\approx 9F_{n-4} + 4F_{n-5} = F_3 F_3 F_{n-4} + F_2 F_2 F_{n-5}$$

$$= i(P_3')i(P_3')i(P_{n-5}) + i(P_2')i(P_2')i(P_{n-6}) .$$

That is, $\min i(T'_3)$ is realized in Figure 3a.

To prove (b), we note that for the tree in Figure 3b, we have $i(T'_3) = 4F_{n-2} + F_{n-3} \approx 12.09016992\cdots F_{n-4}$. On the other hand, for an arbitrary $T_3$ with $n$ large enough, we have $F_{r_1} F_{s_1} F_{t_1} + F_{r_2} F_{s_2} F_{t_2} \approx 13.59764677\cdots \tau^{n-5} \approx 11.61377685\cdots F_{n-4} < 12.09016992\cdots F_{n-4} \approx 4F_{n-2} + F_{n-3}$. That is, $\max i(T'_3)$ is realized in Figure 3b.

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