FACTORIZATION OF k-QUASIHYPONORMAL OPERATORS

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ABSTRACT. Let A be the class of all operators T on a Hilbert space H such that
\( R(T^{k+1}) \subseteq R(T^{k+1}) \), for a positive integer k. It has been shown that if \( T \in A \), there exists a unique operator \( C_T \) on H such that

(i) \( T^{k+1} = T^{k+1}C_T \);

(ii) \( \| C_T \| = \inf \{ \mu : \mu > 0 \text{ and } (T^{k+1}T^{k+1})^* < \mu (T^{k+1}T^{k+1}) \} \);

(iii) \( N(C_T) = N(T^{k+1}) \) and

(iv) \( R(C_T) \subseteq R(T^{k+1}) \).

The main objective of this paper is to characterize k-quotishyponormal; normal, and self-adjoint operators \( T \) in A in terms of \( C_T \). Throughout the paper, unless stated otherwise, H will denote a complex Hilbert space and T an operator on H, i.e., a bounded linear transformation from H into H itself. For an operator T, we write \( R(T) \) and \( N(T) \) to denote the range space and the null space of T.

KEY WORDS AND PHRASES. Self-adjoint, normal, unitary, quasinormal, hyponormal, quasi hyponormal, k-quasi hyponormal, isometry, partial isometry, null space, range space and the projection.

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1. INTRODUCTION

T is said to be quasinormal if \( T(T^*) = (T^*)T \), hyponormal if \( T^*T \geq TT^* \) or equivalently \( \| T^*x \| < \| Tx \| \) for each \( x \) in H, k-quasi hyponormal (Campbell and Gupta
[1]) for a positive integer k if $T^k(T^*T - T_*T)^{k/2} > 0$ or equivalently
\[ ||T^kTx|| < ||T^{k+1}x|| \] for each $x$ in $H$.

The purpose of this paper is to consider the class $A$ of those operators $T$ such that $R(T^k*) \subseteq R(T^{k+1}*)$ for a positive integer $k$. More precisely, our aim is to identify those operators $T$ in $A$ which are $k$-quasihyponormal, normal and self-adjoint. The motivation is due to Embry [2] who considered the class of operators $T$ satisfying $R(T) \subseteq R(T^*)$ and Patel [5] who discussed the class of operators $T$ satisfying $R(T*T) \subseteq R(T^2)$. If $T \in A$, then by Douglas' theorem [3, Theorem 1] there exists a unique operator $C_T$ such that

(i) $T^kT = T^{k+1}C_T$;

(ii) $||C_T||^2 = \inf \{ \mu > 0 \text{ and } (T^kT)(T^kT)^* \leq \mu T^{k+1}T^{k+1} \}$;

(iii) $N(C_T) = N(T^kT)$; and

(iv) $R(C_T) \subseteq R(T^{k+1})$.

2. MAIN RESULTS.

By Douglas' theorem [3, Theorem 1], the class $A$ contains all $k$-quasihyponormal operators.

**Theorem 2.1.** An operator $T$ in $A$ is $k$-quasihyponormal if and only if $||C_T|| < 1$.

**Proof.** If $||C_T|| < 1$, $||T^kT^kx|| = ||C_T T^{k+1}x|| < ||T^{k+1}x||$ for all $x$ in $H$ and hence $T$ is $k$-quasihyponormal.

Conversely, assume that $T$ is $k$-quasihyponormal. Since

$||C_T T^{k+1}x|| = ||T^kT^kx|| < ||T^{k+1}x||$ for all $x$ in $H$, $||C_T^*x|| < ||y||$ for all $y$ in $R(T^{k+1})$. Also since $R(C_T) \subseteq R(T^{k+1})$,

\[ \frac{1}{||C_T||} = \frac{1}{||C_T^*||} \]

i.e. $R(T^{k+1}) \subseteq N(C_T)$, we have $C_T^*x = 0$ for all $x$ in $R(T^{k+1})$. Thus for each $x$ in $H$, $||C_T^*x|| < ||x||$ and consequently $||C_T|| = ||C_T^*|| < 1$.

To prove our next result, we need the following lemma.

**Lemma 2.1.** Let $T$ be a quasinormal operator. Then for any positive integer $k$

(a) $T^kT^k = T^{k-1}T^kT$;

(b) $||((T^kT)^{k/2}x)|| = ||T^kx||$ for all vectors $x$ in $H$;

(c) $N(T^kT) \subseteq N(T^k)$. 

**Proof.** (a) We prove it by induction on $k$. For $k = 1$, trivial. For $k = 2$, again it holds since $T$ is quasinormal. Now assume that the result is true for any positive integer $m > 2$. Then $T^mT^{m+1} = (T^mT)^T = (T^mT^mT)T = T^{m-1}TT^mT = T^mT^mT^mT$. Hence by induction the result follows. (b) It is an immediate consequence of the fact that if $T$ is quasinormal, then $(T^kT)^k = T^kT^k$ for any positive integer $k$. (c)
Let $x \in N(T^* T)$. Then $T^* T x = 0$, i.e., $T^* T T^* T - I x = 0$ by (a). Thus $T^* T - I x \in N(T^* T) = N(T)$. But $N(T) \subseteq N(T^* T)$ since $T$ is quasinormal. Therefore $T^* T - I x = 0$, i.e., $x \in N(T^* T)$.

By using the lemma we obtain the following

**THEOREM 2.2.** Let $T \in A$ be a quasinormal operator. Then $\mathcal{C}_T$ is a quasinormal partial isometry with $R(\mathcal{C}_T) = R(T^* T)$.

**PROOF.** We have $\| (T^* T)^{-1} T x \| = \| T^* T x \| = \| T^* T T^* T x \| = 1$ for any $x \in H$. Thus $\mathcal{C}_T$ is an isometry on $R(T^* T)$. But $R(T^* T) = \mathcal{N}(\mathcal{C}_T)$. Therefore $\mathcal{C}_T$ is a partial isometry. Further, since the initial space of a partial isometry $S$ equals the set of all those vectors $x$ satisfying $\| S x \| = \| x \|$ [4, p. 63] and since $\mathcal{C}_T$ is an isometry on $R(T^* T)$, therefore $R(T^* T) \subseteq N(\mathcal{C}_T)$, the initial space of $\mathcal{C}_T$. Hence $R(T^* T) = N(\mathcal{C}_T)$.

We now prove that $\mathcal{C}_T$ is quasinormal. By making use of Lemma 2.1 again, we see that $N(\mathcal{C}_T) = N(T^* T) \subseteq N(T^* T) \subseteq N(T^* T^* T) = N(\mathcal{C}_T)$ since $R(\mathcal{C}_T) = R(T^* T)$. From this it follows that $N(\mathcal{C}_T) \subseteq N(\mathcal{C}_T)$. Now, if $x \in N(T^* T)$, then $T^* T x = 0$, i.e., $T x \in N(T^* T)$. That means $T^* T x = 0$ or $x \in N(T^* T) = N(T)$. This completes the proof.

**THEOREM 2.3.** An operator $T$ in $A$ is normal if and only if $\mathcal{C}_T$ is a normal partial isometry with $R(\mathcal{C}_T) = R(T)$.

**PROOF.** Let $T$ be normal. Then by Theorem 2.2, $\mathcal{C}_T$ is a partial isometry with $R(\mathcal{C}_T) = R(T^* T)$ and hence $R(\mathcal{C}_T) = R(T^* T)^{-1} = N(T^* T)^{-1} = R(T)$. Thus by Lemma 2.2, $N(\mathcal{C}_T) = N(T^* T) = N(T)$. Therefore $R(\mathcal{C}_T) = R(T^* T) = N(T^* T)^{-1} = N(T)^{-1} = N(C_T) = R(C_T)$. Since $\mathcal{C}_T$ is the projection on $R(\mathcal{C}_T)$ and $\mathcal{C}_T \mathcal{C}_T$ is the projection on $R(\mathcal{C}_T)$, we conclude that $\mathcal{C}_T \mathcal{C}_T = \mathcal{C}_T \mathcal{C}_T$.

Assume on the other hand that $\mathcal{C}_T$ is a normal partial isometry with $R(\mathcal{C}_T) = R(T)$. Since $R(\mathcal{C}_T) \subseteq R(T)$, we have $R(\mathcal{C}_T) = R(T)$
N(T) by Lemma 2.2. Thus \( \|T^*x\| = \|Tx\| \) for each \( x \in R(T^k) \). Further since \( C_T^* \) is a partial isometry on \( R(C_T) = R(T^k) \), we have \( \|T^{k+1}x\| = \|C_T^* T^{k+1}x\| = \|T^{k+1}x\| \) for each \( x \in H \). Thus \( \|T^*y\| = \|Ty\| \) for each \( y \in R(T^k) \). Hence \( \|T^*x\| = \|Tx\| \) for each \( x \in H \), i.e., \( T \) is normal.

**COROLLARY 2.1.** Let \( T \in A \). Then \( T \) is normal and one-to-one if and only if \( C_T \) is a unitary operator with \( R(C_T) = R(T) \).

**PROOF.** Suppose \( T \) is normal and one-to-one. Then by Theorem 2.3, \( C_T \) is a normal partial isometry with \( R(C_T) = R(T) \). Since \( N(C_T) = N(T) = \{0\} \), we have \( N(C_T) = H \) and thus \( C_T \) is an isometry and consequently \( C_T \) is a unitary operator.

Conversely, if \( C_T \) is a unitary operator with \( R(C_T) = R(T) \), \( T \) is normal by Theorem 2.3. Also by Lemma 2.2, \( N(T) = N(T^k) = N(C_T) = \{0\} \), therefore \( T \) is one-to-one.

The next corollary characterizes self-adjoint operators in \( A \).

**COROLLARY 2.2.** Let \( T \in A \). \( T \) is self-adjoint if and only if \( C_T \) is the projection on \( R(T) \).

**PROOF.** Suppose \( T \) is self-adjoint. Then by Theorem 2.3, \( R(C_T) = R(T) = R(T^k) \).

Since \( T^k = T \) and \( T \) is self-adjoint, we have \( T^{k+1} = T^{k+1} C_T \), i.e.,

\[
C_T^{-1} T^{k+1} = T^{k+1}
\]

This means \( C_T^{-1} = I \) on \( R(T^k) = R(T) \). Also \( C_T^* = 0 \) on \( R(T) \) as \( R(T) = R(C_T) \). Therefore \( C_T \) is the projection on \( R(T) \).

Assume now that \( C_T \) is the projection on \( R(T) \). Then \( C_T = R(T) \) and hence by Lemma 2.2, \( N(C_T) = N(T^k) = N(T) \). Also, as in the proof of Theorem 2.3, we have \( N(C_T) = N(C_T) \).

Therefore \( T^k x = Tx \) for all \( x \in R(T^k) \). Moreover \( T^k C_T = T^{k+1} \) implies \( T^k = C_T^{k+1} \) as \( C_T \) is self-adjoint. But \( C_T \) is the projection on \( R(T) = R(T^k) \), therefore \( C_T^{k+1} = T^{k+1} \), that means \( T^k y = Ty \) for all \( y \in R(T^k) \). Thus \( T^k x = Tx \) for all \( x \in H \) or \( T \) is self-adjoint.

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