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RESEARCH EXPOSITORY AND SURVEY ARTICLE

SOME FIXED POINT ITERATION PROCEDURES

B.E. RHOADES

Department of Mathematics
Indiana University
Bloomington, IN 47405 U.S.A.

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ABSTRACT. This paper provides a survey of iteration procedures that have been used to obtain fixed points for maps satisfying a variety of contractive conditions. The author does not claim to provide complete coverage of the literature, and admits to certain biases in the theorems that are cited herein. In spite of these shortcomings, however, this paper should be a useful reference for those persons wishing to become better acquainted with the area.

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1. ITERATION PROCEDURES.

The literature abounds with papers which establish fixed points for maps satisfying a variety of contractive conditions. In most cases the contractive definition is strong enough, not only to guarantee the existence of a unique fixed point, but also to obtain that fixed point by repeated iteration of the function. However, for certain kinds of maps, such as nonexpansive maps, repeated function iteration need not converge to a fixed point. A nonexpansive map satisfies the condition $\|Tx - Ty\| \leq \|x - y\|$ for each pair of points x, y in the space. A simple example is the following. Define $T(x) = 1 - x$ for $0 \leq x \leq 1$. Then T is a nonexpansive selfmap of [0,1] with a unique fixed point at $x = 1/2$, but, if one chooses as a starting point the value $x = a, a \neq 1/2$, then repeated iteration of T yields the sequence $\{1-a, a, 1-a, a, \ldots\}$.

In 1953 W.R. Mann [32] defined the following iteration procedure. Let A be a lower triangular matrix with nonnegative entries and row sums 1. Define $x_{n+1} = T(v_n)$, where

$$v_n = \sum_{k=0}^{n} a_{nk} x_k.$$

The most interesting cases of the Mann iterative process are obtained by choosing matrices A such that $a_{n+1}, k = (1 - a_{n+1,n+1}) a_{nk}, k = 0, 1, \ldots, n; n = 0, 1, 2, \ldots$, and either $a_{nn} = 1$ for all n or $a_{nn} < 1$ for all $n > 0$. Thus, if one chooses any sequence $\{c_n\}$ satisfying (i) $c_0 = 1$, (ii) $0 \leq c_n < 1$ for $n > 0$, and (iii) $\sum c_n = \infty$, then the entries of A become

$$a_{nn} = c_n \ ,$$
$$a_{nk} = c_k \prod_{j=k+1}^{n} (1 - c_j), k < n \ . \tag{1.1}$$

and A is a regular matrix. (A regular matrix is a bounded linear operator on ℓ^∞ such that A is limit preserving for convergent sequences.)

The above representation for A allows one to write the iteration scheme in the following form: $x_{n+1} = (1-c_n)x_n + c_nT(x_n)$.

One example of such matrices is the Cesaro matrix, obtained by choosing $c_n = 1/(n+1)$. Another is $c_n = 1$ for all $n \geq 0$, which corresponds to ordinary function iteration, commonly called Picard iteration.

Pichard iteration of the function $S_{1/2} = (I+T)/2$, is equivalent to the Mann iteration scheme with

$$a_{nk} = \frac{1}{2^n}\binom{n}{k}.$$

This matrix is the Euler matrix of order 1, and the transformation $S_{1/2}$ has been investigated by Edelstein [13] and Krasnoselskii [30]. Krasnoselskii showed that, if X is a uniformly convex Banach space, and T is a nonexpansive selfmap of X, then $S_{1/2}$ converges to a fixed point of T. Edelstein showed that the condition of uniform convexity could be weakened to that of strict convexity.

Pichard iteration of the function $S_\lambda = \lambda I + (1-\lambda)T, 0 < \lambda < 1$, for any function T, homogeneous of degree 1, is equivalent to the Mann iteration scheme with

$$a_{nk} = \binom{n}{k}\lambda^k(1-\lambda)^{n-k}.$$

This matrix is the Euler matrix of order $(1-\lambda)/\lambda$. The iteration of S_λ has been investigated by Browder and Petryshyn [7], Opial [36], and Schaefer [50].

Mann showed that, if T is any continuous selfmap of a closed interval $[a,b]$ with at most one fixed point, then his iteration scheme, with $c_n = 1/(n+1)$, converges to the fixed point of T. Franks and Marzec [15] extended this result to continuous functions possessing more than one fixed point in the interval.

A matrix A is called a weighted mean matrix if A is a lower triangular matrix with nonzero entries $a_{nk} = p_k/P_n$, where $\{p_k\}$ is a nonnegative sequence with p_0 positive and

$$P_n = \sum_{k=0}^{n} p_k \to \infty,$$

The author [42] extended the above-mentioned result of Franks and Marzec to any continuous selfmap of an interval $[a,b]$, and A any weighted mean matrix satisfying the condition

$$\lim_n \sum_{k=0}^{\infty} |a_{nk} - a_{n-1,k}| = 0.$$

In [42] the author also showed that the matrix defined by (1.1) is equivalent to a regular weighted mean matrix with weights

$$p_k = \frac{c_k p_0}{\prod_{j=1}^{k}(1-c_j)}, k > 0.$$

Let E be a Banach space, C a closed convex subset of E, T a continuous selfmap of C. Mann [32] showed that, if either of the sequences $\{x_n\}$ or $\{v_n\}$ converges, then so does the other, and to the same limit, which is a fixed point of T. Dotson [12], extended this result to locally convex Hausdorff linear topological spaces E. Consequently, to use the Mann iterative process on nonexpansive maps, all one needs is to establish the convergence of either $\{x_n\}$ or $\{v_n\}$.

For uniformly convex Banach spaces, the following was obtained independently by Browder [6], Kirk [28], and Gohde [16].

Let C be a closed, bounded, and convex subset of a uniformly convex Banach space, T a nonexpansive selfmap of C. Then T has a fixed point.

Unfortunately the proofs of above theorem are not constructive. A number of papers have been written to obtain some kind of sequential convergence for nonexpansive maps. Most such theorems are valid only under some additional hypothesis, such as compactness, and converge only weakly. Some examples of theorems of this type appear in [8]. Halpern [18] obtained two algorithms for obtaining fixed points for nonexpansive maps on Hilbert spaces. In his dissertation, Humphreys [23] constructed an algorithm which can be applied to obtain fixed points for nonexpansive maps on uniformly convex Banach spaces.

A nonexpansive mapping is said to be asymptotically regular if, for each point x in the space, $\lim(T^{n+1}x - T^n x) = 0$. In 1966 Browder and Petryshyn [7] established the following result.

THEOREM 1. ([7, Theorem 6]). Let X be a Banach space, T a nonexpansive asymptotically regular selfmap of X. Suppose that T has a fixed point, and that $I - T$ maps bounded closed subsets of X into closed subsets of X. Then, for each $x_0 \in X, \{T^n x_0\}$ converges to a fixed point of T in X.

In 1972 Groetsch [17] established the following theorem, which removes the hypothesis that T be asymptotically regular.

THEOREM 2. ([17, Corollary 3]). Suppose T is a nonexpansive selfmap of a closed convex subset E of X which has at least one fixed point. If $I - T$ maps bounded closed subsets of E into closed subsets of E, then the Mann iterative procedure, with $\{c_n\}$ satisfying conditions (i), (ii), and (iv) $\sum c_n(1 - c_n) = \infty$, converges strongly to a fixed point of T.

Ishikawa [25] established the following theorem.

THEOREM 3. Let D be a closed subset of a Banach space X and let T be a nonexpansive map from D into a compact subset of X. Then T has a fixed point in D and the Mann iterative process with $\{c_n\}$ satisfying conditions (i) - (iii), and $0 \leq c_n \leq b < 1$ for all n, converges to a fixed point of T.

For spaces of dimension higher than one, continuity is not adequate to guarantee convergence to a fixed point, either by repeated function iteration, or by some other iteration procedure. Therefore it is necessary to impose some kind of growth condition on the map. If the contractive condition is strong enough, then the map will have a unique fixed point, which can be obtained by repeated iteration of the function. If the contractive condition is slightly weaker, then some other iteration scheme is required. Even if the fixed point can be obtained by function iteration, it is not without interest to determine if other iteration procedures converge to the fixed point.

A generalization of a nonexpansive map with at least one fixed point that of a quasi-nonexpansive map. A function T is a quasi-nonexpansive map if it has at least one fixed point, and, for each fixed point $p, \|Tx - p\| \leq \|x - p\|$. The following is due to Dotson [12].

THEOREM 4. Let E be a strictly convex Banach space, C a closed convex subset of E, T a continuous quasi-nonexpansive selfmap of C such that $T(C) \subset K \subset C$, where K is compact. Let $x_0 \in C$ and consider a Mann iteration process such that $\{c_n\}$ clusters at some point in (0,1). Then the sequences $\{x_n\}, \{v_n\}$ converge strongly to a fixed point of T.

A contractive definition which is included in the class of quasi-contractive maps is the following, due to Zamfirescu [52]. A map satisfies condition Z if, for each pair of points x, y in the space, at least one of the following is true:

(i) $\|Tx - Ty\| \leq \alpha\|x - y\|$, (ii) $\|Tx - Ty\| \leq \beta[\|x - Tx\| + \|y - Ty\|]$, or

(iii) $\|Tx - Ty\| \leq \gamma[\|x - Ty\| + \|y - Tx\|]$,

where α, β, γ are real nonnegative constants satisfying $\alpha < 1, \beta, \gamma < 1/2$. As shown in [52], T has a unique fixed point, which can be obtained by repeated iteration of the function. The following result appears in [42].

THEOREM 5. ([42, Theorem 4]). Let X be a uniformly convex Banach space, E a closed convex subset of X, T a selfmap of E satisfying condition Z. Then the Mann iterative process with $\{c_n\}$ satisfying conditions (i), (ii), and (iv) converges to the fixed point of T.

A generalization of definition Z was made by Ciric [11]. A map satisfies condition C if there exists a constant k satisfying $0 \leq k < 1$ such that, for each pair of points x, y in the space,

$$\|Tx - Ty\| \leq k \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\}.$$

In [42] the author proved the following for Hilbert spaces.

THEOREM 6. ([42, Theorem 7]). Let H be a Hilbert space, T a selfmap of H satisfying condition C. Then the Mann iterative process, with $\{c_n\}$ satisfying conditions (i)-(iii) and $\limsup c_n < 1 - k^2$ converges to the fixed point of T.

Chidume [10] has extended the above result to ℓ^p spaces, $p \geq 2$, under the conditions $k^2(p-1) < 1$ and $\limsup c_n < (p-1)^{-1} - k^2$.

As noted earlier, if T is continuous, then, if the Mann iterative process converges, it must converge to a fixed point of T. If T is not continuous, there is no guarantee that, even if the Mann process converges, it will converge to a fixed point of T. Consider, for example, the map T defined by $T0 = T1 = 0, Tx = 1, 0 < x < 1$. Then T is a selfmap of $[0, 1]$, with a fixed point at $x = 0$. However, the Mann iteration scheme, with $c_n = 1/(n+1), 0 < x_0 < 1$, converges to 1, which is not a fixed point of T.

A map T is said to be strictly-pseudocontractive if there exists a constant k, $0 \leq k < 1$ such that, for all points x, y in the space,

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2.$$

We shall call denote the class of all such maps by P_2. Clearly P_2 mappings contain the nonexpansive mappings, but the classes P_2, C, and quasi-nonexpansive mappings are independent.

THEOREM 7. ([42, Theorem 8]). Let H be a Hilbert space, E a compact, convex subset of H, T a P_2 selfmap of E. Then the Mann iteration scheme with $\{c_n\}$ satisfying conditions (i)-(iii) and $\limsup c_n = c < 1 - k$, converges strongly to a fixed point of T.

A pseudocontractive mapping is a P_2 mapping with $k = 1$. Let P_3 denote the family of pseudocontractive mappings. Hicks and Kubicek [22] gave an example of a P_3 mapping with a fixed point such that the Mann iterative procedure, with $c_n = 1/(n+1)$, does not converge to the fixed point.

Ishikawa [24] defined the following iterative procedure. Given any x_0 in the space, define

$$x_{n+1} = \alpha_n T[\beta_n T x_n + (1 - \beta_n)x_n] + (1 - \alpha_n)x_n,$$

where $\{\alpha_n\}, \{\beta_n\}$ are sequences of positive numbers satisfying the conditions $0 \leq \alpha_n \leq \beta_n \leq 1, \lim \beta_n = 0, \sum \alpha_n \beta_n = \infty$. He then established the following result.

THEOREM 8. Let E be a convex, compact subset of a Hilbert space H, T a lipschitzian P_3 selfmap of E. Then $\{x_n\}$ converges strongly to a fixed point of T.

Qihou [38] extended the above result to lipschitzian hemicontractive maps. A hemicontractive map is a pseudocontractive map with respect to a fixed point; i.e., if p is any fixed point of T, and x is any point in the space, then T satisfies $\|Tx-p\|^2 \leq \|x-p\|^2 + \|x-Tx\|^2$.

Neither the proof of Qihou nor Ishikawa can be used to establish a similar result for the Mann iterative technique.

In 1968 Kannan [27] introduced a contractive definition which did not require continuity of the map. There then followed a spate of fixed point papers dealing with fixed points for a variety of maps which were not necessarily continuous. For example, maps of type C and Z need not be continuous. In an earlier paper the author [45] partially ordered many of these definitions. As noted above, it has been shown that the Mann iteration scheme converges for some of these. For other contractive definitions it is not possible to determine whether or not the Mann or Ishikawa iteration processes converge. However, in some cases it is possible to show that, if they do converge, then they must converge to a fixed point of T. The following theorems illustrate this idea.

THEOREM 9. ([42, Theorems 5,6]). Let X be a Banach space, T a selfmap of X, $T \in C$ or $T \in P_2$. Let $\{c_n\}$ satisfy the conditions (i), (ii), and bounded away from zero. Then, if the corresponding Mann iteration process converges, it converges to a fixed point of T.

THEOREM 10. ([46, Theorem 1]). Let X be a closed convex subset of a normed space, T a selfmap of X satisfying the property that there exist constants $c, k \geq 0, 0 \leq k < 1$ such that, for each $x, y \in X$,

$$\|Tx-Ty\| \leq k\max\{c\|x-y\|, [\|x-Tx\|+\|y-Ty\|], [\|x-Ty\|+\|y-Tx\|]\}.$$

Let $\{c_n\}$ satisfy conditions (i), (ii), and $\lim c_n > 0$. If the Mann iteration scheme converges, then it converges to a fixed point of T.

A contractive definition which is independent of that of Ciric is the following, due to Pal and Maiti [37]. For each pair of points in the space, at least one of the following conditions is satisfied:

(i) $\|x-Tx\|+\|y-Ty\| \leq \alpha\|x-y\|, 1 \leq \alpha < 2$,

(ii) $\|x-Tx\|+\|y-Ty\| \leq \beta\{\|x-Ty\|+\|y-Tx\|+\|x-y\|\}, 1/2 \leq \beta < 2/3$,

(iii) $\|x-Tx\|+\|y-Ty\|+\|Tx-Ty\| \leq \gamma\{\|x-Ty\|+\|y-Tx\|\}, 1 \leq \gamma < 3/2$,

(iv) $\|Tx-Ty\| \leq \delta\max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, [\|x-Ty\|+\|y-Tx\|]/2\}, 0 < \delta < 1$.

Using this definition the author established the following result.

THEOREM 11. ([46, Theorem 2]). Let X be a Banach space, T a selfmap of X satisfying the above contractive definition. Then if the Mann iteration scheme, with $\{c_n\}$ satisfying conditions (i), (ii), and $\lim c_n > 0$, converges, it converges to a fixed point of T.

Similar results have been established for the Ishikawa procedure. In its original form the Ishikawa procedure does not include the Mann iteration process because of the condition $0 \leq \alpha_n \leq \beta_n \leq 1$. For, if the Ishikawa process were to include the Mann process as a special case, then one would have to assign each β_n to be zero, forcing then each α_n to be zero. In an effort to have an Ishikawa type iteration scheme which does include the Mann iterative process as a special case, some authors have modified the inequality condition to read $0 \leq \alpha_n, \beta_n \leq 1$. This change is reflected in the following two theorems.

THEOREM 12. ([35, Theorem 1.2]). Let X be a normed linear space, C a closed convex subset of X. Let T be a selfmap of C satisfying the contractive condition: there exists a constant $k, 0 \leq k < 1$, such that, for each pair of points x, y in X,

$$\|Tx - Ty\| \leq k \max\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\| + \|y - Tx\|\}.$$

If the Ishikawa iteration scheme, with the α_n bounded away from zero, converges to a point p, then p is a fixed point of T.

THEOREM 13. ([40). Let E be a closed convex subset of a Banach space, X, T a selfmap of X satisfying the condition: there exists constants $c, k \geq 0, 0 \leq k < 1$, such that, for each pair of points x, y in X,

$$\|Tx - Ty\| \leq k \max\{c\|x - y\|, [\|x - Tx\| + \|y - Ty\|], [\|x - Ty\| + \|y - Tx\|]\}.$$

If the Ishikawa scheme, with $\{\alpha_n\}, \{\beta_n\}$ satisfying the conditions $0 < a \leq \alpha_n \leq 1, 0 \leq \beta_n \leq 1$ and $\lim \beta_n = 0$, converges to a point p, then p is a fixed point of T.

In some theorems of this type, the contractive definition is weak enough that the a priori existence of a fixed point for T cannot be assured. Thus the convergence of the iteration procedure also yields the existence of a fixed point.

The literature also contains theorems of this type for other contractive definitions. In 1988 the author [47] proved that, for many contractive definitions, even though the function need not be continuous everywhere, it is always continuous in a neighborhood of a fixed point. Therefore, assuming the convergence of the Mann procedure is tantamount, in some cases, to assuming the convergence to a fixed point, in light of the early result of Mann [32].

We now return to nonexpansive maps.

THEOREM 14. ([1]). Let K be a closed bounded convex subset of a Hilbert space H, T a nonexpansive selfmap of K. For each point e of K, the Cesaro transform of $\{T^n e\}$ converges weakly to a fixed point of T.

If T is also an odd map, then Baillon [2] showed that the convergence of the Cesaro transform of $\{T^n e\}$ is strong. He also extended the above theorm to L^p spaces.

We now mention some results for other matrix transforms of iterates of T.

THEOREM 15. ([5, Theorem 1]). Let T be a Hilbert space, C a closed bounded convex subset of H, T a nonexpansive selfmap of C. Suppose that A is an infinite matrix with zero column limits and satisfies $\lim_n \sum_k (a_{n,k+1} - a_{nk})^+ = 0$. Then, for each x in $C, \{\sum a_{nk} T^k x\}$ converges weakly to a fixed point of T.

A sequence x is said to be almost convergent to a limit q if $\lim_n [x_n + x_{n+1} + \cdots + x_{n+p-1}]/n = q$, uniformly in p. An infinite matrix A is said to be strongly regular if A assigns a limit to each almost convergent sequence. Necessary and sufficient conditions for A to be strongly regular were established by Lorentz [31]. These conditions are that A be regular and also satisfy $\lim_n \sum |a_{n,k+1} - a_{nk}| = 0$.

A uniformly convex Banach space X has a modulus of convexity $\delta(\varepsilon)$ defined by

$$\delta(\varepsilon) = \inf\{1 - \|x - y\|/2 : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon\}, 0 < \varepsilon \leq 2.$$

Let $\{u_n\}$ be a bounded sequence in a closed convex subset C of X. Define $r_m(x) = \sup\{\|u_n - x\| : n \geq m\}$, and denote by c_m the unique point in C with the property that $r_m(c_m) = \inf\{r_m(x) : x \in C\}$. Then $\lim c_m = c$, and c is called the asymptotic center of $\{u_n\}$. (See, e.g. [14])

Bruck proved the following theorem.

THEOREM 16. ([9, Theorem 1.1]). Suppose that C is a closed convex subset of a real Hilbert space and T is a nonexpansive selfmap of C with a fixed point. Then for each x in

C, A a strongly regular matrix, the A-transform of $\{T^n x\}$ converges weakly to a fixed point p, which is the asymptotic center of $\{T^n x\}$.

Schoenberg [51] obtained the following characterization for nonexpansive maps on a Hilbert space.

THEOREM 17. ([51]). Let T be a nonexpansive of a Hilbert space H, A be a nonnegative matrix with row sums one and zero column limits,

$$S_a = \sum_{k=0}^{\infty} a_{nk} T^k x.$$

Then the following are equivalent;

(a) $\{S_n\}$ is weakly convergent to the asymptotic center z of $\{T^n x\}$.

(b) $\{S_n\}$ is weakly convergent to a fixed point for T.

(c) If y is a weak limit point of $\{S_n\}$, then $\lim_{n\to\infty}(\mathrm{Re}\langle T^n x - T^{n+1}x, z - Ty\rangle) = 0$

(d) If y is a weak limit point of $\{S_n\}$, then

$$\lim_{n\to\infty}\Big(\sum_{k=0}^{\infty} a_{nk}\mathrm{Re}\langle T^k x - T^{k+1}x, z - Ty\rangle\Big) = 0.$$

(e) Each weak limit point of $\{S_n\}$ is a fixed point of T.

The following theorem establishes a fixed point for a contractive definition which includes nonexpansive mappings.

THEOREM 18. [34]. Let X be a uniformly convex space, K a nonempty closed and uniformly convex subset of X, T a selfmap of K satisfying

$$\begin{aligned} \|Tx - Ty\| \le a(x,y)\|x-y\| + b(x,y)\|x - Tx\| + \\ b'(x,y)\|y - Ty\| + c(x,y)\|x - Ty\| + c'(x,y)\|y - Tx\|, \end{aligned} \tag{1.2}$$

where $a, b, b', c' \ge 0, (a + b + b' + c + c')(x,y) \le 1$, and $b'(x,y) = b(y,x), c'(x,y) = c(x,y)$, for all x, y in X. If

$$\sup_{x,y\in K}(b + c')(x,y) < 1 \tag{1.3}$$

and

$$\inf_{\|x\|\le r}\|x - Tx\| = 0 \text{ for some finite } r \tag{1.4}$$

then T has a fixed point in K.

Define

$$\phi(y) = \sum_{j=0}^{n} a_{nj}\|y - T^j e\|^p, n = 0, 1, 2, \ldots, 1 < p < \infty. \tag{1.5}$$

where A is any nonnegative triangular matrix with row sums one, $y, e, \in K$. For each n define $s_n^{(p)}$ to be the unique point of K where (1.5) assumes its minimum.

THEOREM 19. ([48, Theorem 2]). Let $1 < p < \infty, X = \ell^p, K$ a nonempty closed, bounded, convex subset of X, T an asymptotically regular selfmap of K satisfying (1.2) and (1.3) with $b = b'$. Then each weak limit point of $\{s_n^{(p)}\}$ is a fixed point of T.

THEOREM 20. ([48, Theorem 3]). Let $1 < p < \infty, X = \ell p, K$ a nonempty closed, bounded, convex subset of X, T a quasi-nonexpansive selfmap of K. If each subsequential weak limit of $\{s_n^{(p)}\}$ is a fixed point of T, then $\{s_n^{(p)}\}$ converges weakly to a fixed point of T.

The special case of the above theorems in which A is the Cesaro matrix and T is nonexpansive appear in Beauzamy and Enflo [4].

We shall now examine two other iteration methods for obtaining fixed points. The first of these is due to Kirk [29] and is defined as follows. Let $\{x_n\}$ be any sequence of points in the space. Then the iteration procedure is defined by the operator

$$S = \sum_{i=0}^{k} \alpha_i T^i, \tag{1.6}$$

where k is a fixed integer, $k \geq 1, a_i \geq 0, i = 0, 1, 2, \ldots, a_1 > 0$ and $\sum_{i=0}^{k} \alpha_i = 1$.
For T any nonexpansive selfmap of a convex set, S and T have the same fixed points. (See [29].)

Let T be a selfmap of a Banach space X satisfying, for each x, y in X,

$$\|Tx - Ty\| \leq \max\{\|x - y\|, [\|x - Tx\| + \|y - Ty\|]/2, [\|x - Ty\| + \|y - Tx\|]/2\}. \tag{1.7}$$

In [41] it is shown that, if S satisfies (1.6), then S and T have the same fixed points. The following result is then proved.

THEOREM 21. ([41, Theorem 2]). Let X be a uniformly convex Banach space, K a bounded closed convex subset of X, T a selfmap of K which satisfies condition (7), S defined by (6). If $I - S$ maps bounded closed subsets of K into closed sets, then, for each $x_0 \in K$ the sequence $\{S^n x_0\}$ converges to a fixed point of T in K.

The other fixed point iteration procedure is defined by

$$S = \sum_{i=0}^{\infty} \alpha_i T^i, \sum_{i=0}^{\infty} \alpha_i = 1, \alpha_k \alpha_{k+1} \neq 0 \text{ for at least one integer } k. \tag{1.8}$$

Massa [33] has shown that, if T is quasi-nonexpansive, then S and T have the same fixed points. He has also proved the following.

THEOREM 22. ([33, Theorem 2]). Let K be a closed convex subset of a uniformly convex Banach space X, T a selfmap of K satisfying

$$\|Tx - Ty\| \leq a(x, y)\|x - y\| + b(x, y)(\|x - Tx\| + \|y - Ty\|) + c(x, y)(\|x - Ty\| + \|y - Tx\|),$$

where $a, b, c, \geq 0, (a + 2b + 2c)(x, y) \leq 1, \inf_{x, y \in K} b(x, y) > 0$, and T has at least one fixed point. Then $\{S^n x\}$ converges to the only fixed point of T.

2. RATE OF CONVERGENCE.

There has been no systematic study of the rate of convergence for these iteration procedures, and it is doubtful if any global statements can be made, since there is nothing about these iteration procedures to cause their analysis to be different from that of other approximation methods.

The author [44] obtained some evidence on the behavior of the Mann iteration procedure for the decreasing functions $f(x) = 1 - x^m, g(x) = (1 - x)^m$ for $1 \leq m \leq 6$ and $c_n = [(n+1)(n+2)]^{-1/k}, 3 \leq k \leq 8$. The fixed point of each function was first found by the bisection method, accurate to 10 places. Then both the Mann iteration and Newton-Raphson methods were employed to find each fixed point to within 8 places, using initial guesses of $x_0 = .1, .2, \ldots, .9$. The author [43] also examined the functions g for $m = 7, 8, \ldots, 29$, with an initial guess of $x_0 = .9$, and $\alpha_n = \beta_n = (n+1)^{-1/2}$. In each case the Mann iterative procedure converged to a fixed point (accurate to eight places) in 9-12 iterations, whereas the Ishikawa method required from 38 to 42 iterations for the same degree of accuracy. For increasing functions the Ishikawa method is better than the Mann process, but ordinary function iteration is the best of all three for increasing functions.

An examination of the printout showed that the Newton-Raphson method converges faster than the Mann scheme. This is not surprising, since the Newton-Raphson method converges quadratically, whereas the Mann process converges linearly. However, whereas the rate of convergence of the Newton-Raphson method is very sensitive to the starting point, the rate of convergence for the Mann process appears to be independent of the initial guess.

3. STABILITY.

We shall now discuss the question of stability of iteration processes, adopting the definition of stability that appears in [19].

Let X be a Banach space, T a selfmap of X, and assume that $x_{n+1} = f(T, x_n)$ defines some iteration procedure involving T. For example, $f(T, x_n) = Tx_n$. Suppose that $\{x_n\}$ converges to a fixed point p of T. Let $\{y_n\}$be an arbitrary sequence in X and define $\varepsilon_n = \|y_{n+1} - f(T, y_n)\|$ for $n = 0, 1, 2, \ldots$. If $\lim_n = 0$ implies that $\lim_n y_n = p$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable. The first result on T-stable mappings was proved by Ostrovski for the Banach contraction principle. In [20] the authors show that function iteration is stable for a variety of contractive definitions. Their best result for function iteration is the following.

THEOREM 23. ([20, Theorem 2]). Let X be a complete metric space, T a selfmap of X satisfying the contractive condition of Zamfirescu. Let p be the fixed point of T. Let $x_0 \in K$, set $x_{n+1} = Tx_n, n \geq 0$. Let $\{y_n\}$ be a sequence in X, and set $\varepsilon_n = d(y_{n+1}, Ty_n)$ for $n = 0, 1, 2, \ldots$. Then

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} 2\delta^{n+1-k} d(x_k, x_{k+1}) + \delta^{n+1} d(x_0, y_0) + \sum_{k-0}^{n} \delta^{n-k} \varepsilon_k, n = 0, 1, 2, \ldots$$

where

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$$

and

$$\lim_{n\to\infty} y_n = p \text{ if and only if } \lim_{n\to\infty} \varepsilon_n = 0.$$

For the Mann iteration procedure their best result is the following.

THEOREM 24. ([20, Theorem 3]). Let $(X, \|\cdot\|)$ be a normed linear space, T a selfmap of X satisfying the Zamfirescu condition. Let $x_0 \in X$, and suppose that there exists a fixed point p and $x_n \to p$, where $\{x_n\}$ denotes the Mann iterative procedures with the $\{c_n\}$ satisfying (i), (ii), and $0 < a \leq c_n \leq b < 1$. Suppose $\{y_n\}$ is a sequence in X and $\varepsilon_n = \|y_{n+1} - [(1-c_n)y_n + c_n Ty_n]\|$ for $n = 0, 1, 2, \ldots$. Then

$$\|p - y_{n+1}\| \leq \|p - x_{n+1}\| + (1 - a + a\delta)^{n+1}\|x_0 - y_0\| +$$
$$\sum_{i=0}^{n} 2\delta b(1 - a + a\delta)^{n+1}\|x_i - Tx_i\| + \sum_{i=0}^{n}(1 - a + a\delta)^{n-i}\varepsilon_i \quad n = 0, 1, 2, \ldots,$$

where

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$$

and

$$\lim_{n\to\infty} y_n = p \text{ if and only if } \lim_{n\to\infty} \varepsilon_n = 0.$$

For the iteration method of Kirk [29], they have the following result.

THEOREM 25. ([20, Theorem 4]). Let $(X, \|\cdot\|)$ be a Banach space, T a selfmap of X satisfying the Banach contractive condition $\|Tx - Ty\| \leq c\|x - y\|$ for some constant $c, 0 \leq c < 1$. Let p be the fixed point of T, x_0 an arbitrary point of X. Set

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i T^i x_n \text{ for } n = 0, 1, 2, \ldots,$$

where k is an integer such that $k \geq 1, a_i \geq 0$ for $i = 0, 1, 2, \ldots, k, a_1 > 0$, and

$$\varepsilon_n = \|y_n - \sum_{i=0}^{k} \alpha_i T^i y_n\| \text{ for } n = 0, 1, 2, \ldots.$$

Then

$$\|p - y_{n+1}\| \leq \|p - x_{n+1}\| + \Big(\sum_{i=0}^{k} \alpha_j c_j\Big)^{n+1}\|x_0 - y_0\| + \sum_{i=0}^{n}\Big(\sum_{j=0}^{k} \alpha_j c^j\Big)^{n-i}\varepsilon_i, \quad n = 0, 1, 2, \ldots$$

and

$$\lim_{n\to\infty} y_n = p \text{ if and only if } \lim_{n\to\infty} \varepsilon_n = 0.$$

In a recent paper the author [49] has proved each of the above theorems for a contractive definition independent of that of Zamfirescu.

Consider the following contractive condition: there exists a constant c satisfying $0 \leq c < 1$ such that, for each pair of points x, y in X,

$$\|Tx - Ty\| \leq c\max\{\|x - y\|, \|x - Ty\|, \|y - Tx\|\}. \tag{3.1}$$

Our first result is the following.

THEOREM 26. ([49, Theorem 1]). Let (X, d) be a complete metric space, T a selfmap of X satisfying (3.1). Let p be the fixed point of T. Let $x_0 \in X$ and define $x_{n+1} = Tx_n$. Let $\{y_n\} \subset X$. Define $\varepsilon_n = d(y_{n+1}, Ty_n)$. Then

$$d(p, y_{n+1}) \leq d(p, x_{n+1}) + \sum_{k=0}^{n} mc^{n+1-k} d(x_k, x_{k+1}) + c^{n+1} d(x_0, y_0) + \sum_{k=0}^{n} c^{n-k}\varepsilon_k,$$

where $m = 1/(1-c)$, and $\lim_{n\to\infty} = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

THEOREM 27. ([49, Theorem 2]). Let $(X, \|\cdot\|)$ be a Banach space, T a selfmap of X satisfying (3.1), p the fixed point of T. Let $\{x_n\}$ denote the Mann iterative process with the $\{c_n\}$ satisfying (i) - (iii),

$$\lim_n \prod_{i=1}^{n} (1 - c_i + cc_i) = 0,$$

and

$$\sum_{j=0}^{n} \prod_{i=j+1}^{n} (1 - c_i + cc_i) \text{ converges }.$$

Let $\{y_n\} \subset X$ and define $\varepsilon_n = \|y_{n+1} - (1-c_n)y_n - c_n T y_n\|$. Then

$$\|p - y_{n+1}\| \le \|p - x_{n+1}\| + cm \sum_{i=0}^{n} \prod_{i=j+1}^{n} (1 - c_i + cc_i)\|x_i - Tx_i\| +$$

$$\prod_{i=0}^{n} (1 - c_i + cc_i)\|x_0 - y_0\| + \sum_{j=0}^{n} \prod_{i=j+1}^{n} (1 - c_i + cc_i)\varepsilon_j,$$

where it is understood that the product is 1 when $j = n$. Then

$$\lim_{n\to\infty} y_n = p \text{ if and only if } \lim_{n\to\infty} \varepsilon_n = 0.$$

The stability result for Kirk iterations has been extended to the two independent contractive definitions mentioned earlier.

THEOREM 28. ([49, Theorem 3]). Let X be a Banach space, T a selfmap of X satisfying condition (3.1). Let p be the fixed point of $T, x_0 \in X$, and define $\{x_{n+1}\}$ as in (1.6). Let $\{y_n\} \subset X$, and define

$$\varepsilon_n = \|y_n - \sum_{i=0}^{k} \alpha_i T^i y_n\| \text{ for } n = 0, 1, 2, \ldots.$$

Then, with $m = 1/(1-c)$,

$$\|p - y_{n+1}\| \le \|p - x_{n+1}\| + \sum_{j=0}^{n} \Big(\sum_{i=0}^{k} \alpha_i c^i\Big)^j \Big(\sum_{i=0}^{k} \alpha_i c^{i-1}[(1-c)^{-i} - 1]\Big) \|x_{n-j} - Tx_{n-j}\|$$

$$+ \Big(\sum_{i=0}^{k} \alpha_i c^i\Big)^{n+1} \|x_0 - y_0\| + \sum_{j=0}^{n} \Big(\sum_{i=0}^{k} \alpha_i c^i\Big)^{n-j} \varepsilon_j,$$

and $\lim_{n\to\infty} = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

THEOREM 29. ([49, Theorem 4]). Let X be a Banach space, T a selfmap of X satisfying condition Z. Let p be the fixed point of $T, x_0 \in X$, and define $\{x_{n+1}\}$ as in (1.6). Let $\{y_n\} \subset X$, and define

$$\varepsilon_n = \|y_n - \sum_{i=0}^{k} \alpha_i T^i y_n\| \text{ for } n = 0, 1, 2, \ldots.$$

Then, with

$$\delta = \max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\},$$

$$\|p - y_{n+1}\| \le \|p - x_{n+1}\| + \sum_{j=0}^{n}\Big(\sum_{i=0}^{k}\alpha_i\delta^i\Big)^j\Big(\sum_{i=0}^{k}\alpha_i\delta^i\Big)\|x_{n-j} - Tx_{n-j}\| +$$
$$\Big(\sum_{i=0}^{k}\alpha_i\delta^i\Big)^{n+1}\|x_0 - y_0\| + \sum_{j=0}^{n}\Big(\sum_{i=0}^{k}\alpha_i\delta^i\Big)^j\varepsilon_{n-j},$$

vand $\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0$.

We shall now prove that the Ishikawa iteration procedure is T-stable for maps satisfying either condition Z or (3.1).

THEOREM 30. Let $(X, \|\cdot\|)$ be a Banach space, T a selfmap of X satisfying (3.1). Let $\{x_n\}$ be defined by the Ishikawa process; i.e.,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tz_n, z_n = (1-\beta_n)x_n + \beta_n Tx_n, \text{ where } 0 \le \beta_n \le 1, 0 < a \le \alpha_n \le 1$$

for all n, and the $\{\alpha_n\}$ satisfy $\lim_n \prod_{i=1}^{n}(1-\alpha_i + c\alpha_i) = 0$ and $\sum_{j=0}^{n}\prod_{i=j+1}^{n}(1-\alpha_i+c\alpha_i)$ converges.

Let $\{y_n\} \subset X$, and define

$$\varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T[(1-\beta_n)y_n + \beta_n Ty_n]\|.$$

Then

$$\|y_{n+1} - p\| \le \|x_{n+1} - p\| + \prod_{i=0}^{n}(1-\alpha_i + c\alpha_i)\|x_0 - y_0\| + m\alpha_n\|x_n - Tx_n\| +$$
$$\sum_{i=0}^{n-1} m\alpha_{n-i}\prod_{j=i+1}^{n}(1-\alpha_j + c\alpha_j)\|x_i - Tx_i\| + m\alpha_n\|z_n - Tz_n\| +$$
$$\sum_{i=0}^{n-1} m\alpha_{n-i}\prod_{j=i+1}^{n}(1-\alpha_j + c\alpha_j)\|z_i - Tz_i\| + \varepsilon_n +$$
$$\sum_{i=0}^{n-1}\prod_{j=i+1}^{n}(1-\alpha_j + c\alpha_j)\varepsilon_i, \tag{3.2}$$

where $m = c/(1-c)$, and $\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty}\varepsilon_n = 0$.
Proof. Let p be the fixed point of T. Then

$$\|x_{n+1} - p\| \le \|(1-\alpha_n)x_n + \alpha_n Tz_n - p\| \le (1-\alpha_n)\|x_n - p\| + \alpha_n\|Tz_n - p\|.$$
$$\|Tz_n - Tp\| \le c\max\{\|z_n - p\|, \|z_n - Tp\|, \|p - Tz_n\|\} = c\|z_n - p\|.$$

But

$$\|z_n - p\| = \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \le (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\|$$
$$\le (1 - \beta_n + c\beta_n)\|x_n - p\| \le \|x_n - p\|.$$

Therefore $\|x_{n+1} - p\| \le (1 - \alpha_n + c\alpha_n)\|x_n - p\|$, and $\{\|x_n - p\|\}$ is monotone decreasing in n, and so converges to a limit $d \ge 0$. If $d > 0$, then, taking the limit as $n \to \infty$, in the above inequality, yields $d \le (1 - a(1-c))d < d$, a contradiction. Therefore $d = 0$ and $\{x_n\}$ converges to p. Using (3.1), $\|x_n - Tx_n\| \le \|x_n - p\| + \|Tp - Tx_n\| \le (1+c)\|x_n - p\|$, and $\lim_n \|x_n - Tx_n\| = 0$. Note also that $x_n \to p$ implies $z_n \to p$, and hence $\|z_n - Tz_n\| \to 0$.

For any x, y in X, $\|Tx - Ty\| \le m\|x - Tx\| + c\|x - y\|$, where $m = c/(1-c)$. Thus

$$\|y_{n+1} - p\| \le \|x_{n+1} - p\| + \|(1-\alpha_n)x_n + \alpha_n Tz_n - (1-\alpha_n)y_n - \alpha_n T[(1-\beta_n)y_n + \beta_n Ty_n]\| + \varepsilon_n$$
$$\le \|x_{n+1} - p\| + (1-\alpha_n)\|x_n - y_n\| + \alpha_n m\|z_n - Tz_n\|$$
$$+ \alpha_n c\|z_n - (1-\beta_n)y_n - \beta_n Ty_n\| + \varepsilon_n.$$

But

$$\|z_n - (1-\beta_n)y_n - \beta_n Ty_n\| \le (1-\beta_n)\|x_n - y_n\| + \beta_n\|Tx_n - Ty_n\|$$
$$\le (1-\beta_n)\|x_n - y_n\| + \beta_n[m\|x_n - Tx_n\| + c\|x_n - y_n\|$$
$$\le \|x_n - y_n\| + m\|x_n - Tx_n\|,$$

so

$$\|y_{n+1} - p\| \le \|x_{n+1} - p\| + (1 - \alpha_n + c\alpha_n)\|x_n - y_n\| + \alpha_n m\|z_n - Tz_n\| + \alpha_n m\|x_n - Tx_n\| + \varepsilon_n,$$

and (3.2) follows by induction.

An infinite matrix A is called multiplicative if the A-limit of a convergent sequence is equal to some constant multiple of the limit of the sequence. If the matrix has zero column limits and finite norm, then the multiplier is the limit of the row sums.

Returning to the proof of the theorem, suppose that $\varepsilon_n \to 0$. Let A denote the lower triangular matrix with nonzero entries $a_{nn} = m\alpha_0$,

$$a_{nk} = m\alpha_{n-k} \prod_{j=k+1}^{n} (1 - \alpha_j + c\alpha_j), k < n.$$

Then, since $\|x_n - Tx_n\| \to 0$, $\|z_n - Tz_n\| \to 0$, and A is multiplicative, the right hand side of (3.2) tends to zero as $n \to \infty$, and $y_n \to p$.

Suppose $y_n \to p$.

$$0 \le \varepsilon_n = \|y_{n+1} - (1-\alpha_n)y_n - \alpha_n T[(1-\beta_n)y_n + \beta_n Ty_n]\|$$
$$\le \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|T[(1-\beta_n)y_n + \beta_n Ty_n] - Tp\|$$
$$\le \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\{c(1-\beta_n)\|y_n - p\| + c\beta_n\|Ty_n - Tp\|\}$$
$$\le \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\{(1-\beta_n)\|y_n - p\| + c\beta_n\|y_n - p\|\}$$
$$\le \|y_{n+1} - p\| + (1-\alpha_n)\|y_n - p\| + \alpha_n\|y_n - p\|$$
$$\le \|y_{n+1} - p\| + \|y_n - p\| \to 0.$$

THEOREM 31. Let $(X, \|\cdot\|)$ be a Banach space, T a selfmap of X satisfying condition Z. Let $\{x_n\}, \{y_n\}, \{\varepsilon_n\}, \{\alpha_n\}, \{\beta_n\}$ be as in Theorem 30. Then

$$\begin{aligned}\|y_{n+1} - p\| \leq \|x_{n+1} - p\| + \prod_{i=0}^{n}(1 - \alpha_i + \delta\alpha_i)\|x_0 - y_0\| + 2\delta\alpha_n\|x_n - Tx_n\| + \\ \sum_{i=0}^{n-1} 2\delta\alpha_{n-i} \prod_{j=i+1}^{n} (1 - \alpha_j + \delta\alpha_j)\|x_i - Tx_i\| + 2\delta\alpha_n\|z_n - Tz_n\| + \\ \sum_{i=0}^{n-1} 2\delta\alpha_{n-i} \prod_{j=i+1}^{n} (1 - \alpha_j + \delta\alpha_j)\|z_i - Tz_i\| + \varepsilon_n + \\ \sum_{i=0}^{n-1} \prod_{j=i+1}^{n} (1 - \alpha_j + \delta\alpha_j)\varepsilon_i,\end{aligned} \tag{3.3}$$

and $\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

The proof is similar to that of Theorem 1 and will therefore be omitted.

The following establishes a stability theorem for Massa's iteration procedure.

THEOREM 32. ([49, Theorem 5]). Let T satisfy the Banach contraction principle. Let $x_0 \in X$, and define $x_{n+1} = Sx_n$, where S satisfies (1.8). Let $\{y_n\} \subset X$, and define

$$\varepsilon_n = \|y_{n+1} - \sum_{i=0}^{\infty} \alpha_i T^i y_n\|.$$

Then

$$\|y_{n+1} - p\| \leq \|x_{n+1} - p\| + \Big(\sum_{i=0}^{\infty} \alpha_i c^i\Big)^{n+1} \|x_0 - y_0\| + \sum_{j=0}^{n} \Big(\sum_{i=0}^{\infty} \alpha_i c^i\Big)^{n-j} \varepsilon_j,$$

and $\lim_{n\to\infty} y_n = p$ if and only if $\lim_{n\to\infty} \varepsilon_n = 0$.

We conclude with a result on stability for nonexpansive maps. D. de Figureiredo [26, p. 230] established the following result.

THEOREM 33. Let H be a Hilbert space and C a closed, bounded, and convex subset in H containing 0. If T is any nonexpansive selfmap of C, then, for any x_0 in C, the sequence $\{x_n\}$ defined by

$$x_n = T_n^{n^2} x_{n-1}, n = 1, 2, \ldots \text{and } T_n x = \frac{n}{n+1} Tx,$$

converges strongly to a fixed point of T.

THEOREM 34. ([19, Theorem 2]). Let $x_0 \in X$, where $(X, \|\cdot\|)$ is a normed linear space. Suppose that T is a nonexpansive selfmap of X. Let

$$f(T, x_n) = x_{n+1} = T_{n+1}^{(n+1)^2} x_n, n = 0, 1, 2, \ldots \text{where } T_{n+1}x = \frac{n+1}{n+2} Tx.$$

Then $f(T, x_n)$ is T-stable.

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