ON THE THREE-DIMENSIONAL CR-SUBMANIFOLDS OF THE SIX-DIMENSIONAL SPHERE

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(Received December 13, 1989 and in revised form March 27, 1990)

ABSTRACT. We show that the six-dimensional sphere does not admit three-dimensional totally umbilical proper CR-submanifolds.

KEY WORDS AND PHRASES. Totally umbilical submanifolds, totally real submanifolds, CR-submanifolds, almost complex structure.

1980 AMS SUBJECT CLASSIFICATION CODE. 53C40; 53C55.

1. INTRODUCTION. The six-dimensional unit sphere $S^6(1)$ has a nearly Kaehler structure $J$ constructed in a natural way by making use of Cayley division algebra [3]. It is because of this nearly Kaehler, non-Kaehler structure, that $S^6(1)$ has drawn the attention. In particular, almost complex submanifolds, CR-submanifolds and totally real submanifolds of $S^6(1)$ have been considered by A. Gray [4], K. Sekigawa and N. Ejiri [2]. For three-dimensional totally real submanifolds of $S^6(1)$ of constant curvature, N. Ejiri proved the following [2].

THEOREM 1. Let $M$ be a 3-dimensional totally real submanifold of constant curvature $c$ in $S^6(1)$. Then $c = 1$ (totally geodesic) or $c = \frac{1}{10}$ (minimal).

In this paper we consider 3-dimensional CR-submanifolds of $S^6(1)$. We prove the following result:

THEOREM 2. There are no 3-dimensional totally umbilical proper CR-submanifolds in $S^6(1)$.

2. PRELIMINARIES.

Let $C_+$ be the set of all purely imaginary Cayley numbers. The $C_+$ can be viewed as a 7-dimensional linear subspace $\mathbb{R}^7$ of $\mathbb{R}^8$. Consider the unit hypersphere which is centered at the origin

$$S^6(1) = \{x \in C_+ | <x,x> = 1\}.$$ 

The tangent space $T_xS^6$ of $S^6(1)$ at a point $x$ may be identified with the affine subspace of $C_+$ which is orthogonal to $x$. On $S^6(1)$ define a (1,1)-tensor field $J$ by putting

$$J_xU = x \times U,$$

where the above product is defined as in [3] for $x \in S^6(1)$ and $U \in T_xS^6$. 
The above tensor field J determines an almost complex structure (i.e., $J^2 = -I_d$) on $S^6(1)$. The compact simple Lie group of automorphisms $G_2$ acts transitively on $S^6(1)$ and preserves both $J$ and the standard metric on $S^6(1)$, [3].

Now let $G$ be the $(2,1)$-tensor field on $S^6(1)$ defined by

$$G(X, Y) = (\nabla_X J)Y$$

where $\nabla$ is the Levi-Civita connection on $S^6(1)$ and $X, Y \in T_xS^6$.

Since $\nabla_X J$ is skew-symmetric with respect to the Hermitian metric $g$ on $S^6(1)$, it follows that $G$ has the following property

$$g(G(X, Y), Z) + g(G(X, Z), Y) = 0$$

where $X, Y, Z \in \mathfrak{X}(S^6)$.

A submanifold $M$ of of dim $(2p + q)$ in $S^6(1)$ is called a CR-submanifold if there exists a pair of orthogonal complementary distributions $D$ and $\overline{D}$ such that $J\mathcal{D} = D$ and $J\overline{D} \subset \nu$, where $\nu$ is the normal bundle of $M$ and dim $D = 2p$, dim $\overline{D} = q[1]$. Thus the normal bundle $\nu$ splits as $\nu = J\mathcal{D} \oplus \mu$, where $\mu$ is invariant sub-bundle of $\nu$ under $J$.

A CR-submanifold is said to be proper if neither $D = 0$ nor $\overline{D} = 0$.

We denote by $\nabla$, $\nabla$, $\nabla$ the Riemannian connections on $M$, $S^6$ and the normal bundle, respectively. They are related by Gauss formula and Weingarten formula:

$$\nabla_X Y = \nabla_X Y + h(X, Y)$$

$$\nabla_X N = -A_N X + \nabla_X N \quad N \in \nu$$

where $h(X, Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X, Y), N) = g(A_N X, Y)$$

$X$ and $Y$ are vector fields on $M$.

Now a CR-submanifold is said to be totally umbilical if $h(X, Y) = g(X, Y)H$ where $H = \frac{1}{n}(\text{trace } h)$ is the mean curvature vector. If $M$ is a totally umbilical CR-submanifold, then equations (2) and (3) become

$$\nabla_X Y = \nabla_X Y + g(X, Y)H$$

$$\nabla_X N = -g(H, N)X + \nabla_X N$$

Let $R$ be the curvature tensor associated with $\nabla$. Then the equation of Gauss is given by

$$R(X, Y; Z, W) = g(X, Z)g(Y, W) - g(Y, Z)g(X, W)$$

$$+ g(h(X, Z), h(Y, W)) - g(h(Y, Z), h(X, W))$$

It is known that for $X, Y$ in $D$, $G(X, Y) = 0$, and $G(W, W) = 0$ for all $W \in \mathfrak{X}(S^6)$.

3. 3-DIMENSIONAL CR-SUBMANIFOLDS OF $S^6(1)$:

Let $M$ be a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$. Since $M$ is proper, $D \neq \{0\}$ and $\overline{D} \neq \{0\}$. Then since dim $M = 3$, we have dim $D = 2$ and dim $\overline{D} = 1$.

We have the following:

**Lemma 1.** If $M$ is a 3-dimensional totally umbilical proper CR-submanifold of $S^6(1)$, then $H \in JD$.

**Proof.** For $X, Y \neq 0$ in $D$ we use equation (2.5) and the equation $J \nabla_X Y = \nabla_X JY$ to get
\[ J \nabla_X Y + g(X, Y)JH = \nabla_X Y + g(X, Y)H. \] (3.1)

Taking inner product in (3.1) with \( N \in \mu \) we have

\[ g(X, Y)g(JH, N) = g(X, JY)g(H, N) \] (3.2)

In particular, if we let \( Y = JX \) in (3.2) we get

\[ X g(N, N) = 0 \]

From which it follows that \( H \in J \tilde{D} \).

**LEMMA 2.** If \( M \) is a 3-dimensional totally umbilical CR-submanifold of \( S^6(1) \), the \( \| H \| \) is constant.

**PROOF.** Using (2.7) and the equation \( h(X, Y) = g(X, Y)H \) we get

\[ R(X, Y; Z, W) = (1 + H^2) \{ g(X, Z)g(Y, W) \}

\[ - g(Y, Z)g(X, W) \} \] (3.3)

Then since \( \dim M = 3 \), we invoke Schur’s theorem to conclude that \( (1 + H^2) \) is constant. Thus \( \| H \| \) is constant.

4. **PROOF OF THEOREM 2.**

In this section let \( \{ X, JX, Z \} \) denote an orthonormal frame field for the 3-dimensional totally umbilical CR-submanifold \( M \) of \( S^6(1) \). The unit vector fields \( X, JX \) are in \( \mathcal{D} \) and the unit vector field \( Z \) is in \( \tilde{D} \). Since \( M \) is totally umbilical, the equation \( h(X, Y) = g(X, Y)H \) implies that

\[ h(X, JX) = h(X, Z) = h(JX, Z) = 0 \]

and

\[ h(X, X) = h(JX, JX) = h(Z, Z) = H \]

We know from the previous Lemma that \( H \in J \tilde{D} \). Since \( \dim \tilde{D} = 1 \), then one can write \( H = \alpha JZ \) for some smooth function \( \alpha \) on \( M \). Therefore

\[ h(X, X) = h(JX, JX) = h(Z, Z) = \alpha JZ \]

Using equation (2.4) with \( N = JZ \) we get

\[ A_{JZ}X = \alpha X, \quad A_{JZ}JX = \alpha JX, \quad A_{JZ}Z = \alpha Z \] (4.2)

So the frame field \( \{ X, JX, Z \} \) diagonalizes \( A \). Now in \( S^6(1) \) we have equation (2.1) i.e. \( g((\bar{\nabla}_X J)Y, Z) + g((\bar{\nabla}_X J)Z, Y) = 0 \) for any \( X, Y, Z \in \mathcal{D}(S^6) \). Since for \( X, Y \in \mathcal{D} (\bar{\nabla}_X J)Y = 0 \), then using this equation with \( Y = JX \) for our orthonormal frame field \( \{ X, JX, Z \} \) in \( M \), we get

\[ g((\bar{\nabla}_X J)Z, JX) = 0 \] (4.3)

Using equation (2.5), (4.3) and (2.6) with the fact that \( H \in J \tilde{D} \) and \( (\bar{\nabla}_X J)Z = \bar{\nabla}_X JZ - J \bar{\nabla}_X Z \) we get

\[ g(\nabla_X Z, X) = 0 \] (4.4)

Again using equation (2.5) and (2.6) in equation (2.1) with \( Y = X \), we get

\[ g(\nabla_X Z, JX) = \alpha \] (4.5)
Also using equation (2.1) and \((\nabla_{JX}J)Z = \nabla_{JX}JZ - J \nabla_{JX}Z\) we get
\[ g(\nabla_{JX}Z, X) = -\alpha \quad (4.6) \]
Switching the role of \(X\) and \(Y\) in equation (2.1) and letting \(Y = JX\) we obtain
\[ g(\nabla_{JX}Z, JX) = 0 \quad (4.7) \]
Now using the equation \(g((\nabla_XJ)X, JZ) = 0\) and \(g(\nabla_{JX}X, JZ) = 0\) we get
\[ g(\nabla_XX, Z) = 0, \quad g(\nabla_{JX}JX, JZ) = 0 \quad (4.8) \]
From the equation \((\nabla_ZJ)Z = 0\), using equation (4.1) and (4.2) and the fact that \(\nabla Z \in \mathcal{D}\), we get
\[ \nabla Z = 0, \quad \nabla ZJZ = 0 \quad (4.9) \]
Using equations (4.5), (4.6), (4.7), (4.8) and the first part of equation (4.9) we can write the local equations for the frame field \(\{X, JX, Z\}\) as follows:
\begin{align*}
\nabla_XZ &= aJX, \quad \nabla_{JX}Z = -aX, \quad \nabla_ZZ = 0 \\
\nabla_XX &= aJX, \quad \nabla_{JX}X = -bJX + aZ, \quad \nabla_ZX = cJX \\
\nabla_XJX &= -aX - aZ, \quad \nabla_{JX}JX = bX, \quad \nabla_ZJX = -cX \quad (4.10)
\end{align*}
for some smooth functions \(a, b\) and \(c\).

The curvature tensor \(R\) is given by
\[ R(X, Y; Z, W) = \left< \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla Z, W \right> \]
Then using this equation with the help of equations (4.10) we get \(R(X, Z, Z, X) = \alpha^2\), \(\alpha = \|H\|\). But from equation (3.3) we know that \(R(X, Z, Z, X) = -(1 + \alpha^2)\). This is a contradiction and hence \(S^6(1)\) cannot admit a 3-dimensional totally umbilical proper CR-submanifolds.

REFERENCES