CHARACTERIZATION OF HANKEL TRANSFORMABLE GENERALIZED FUNCTIONS

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ABSTRACT. In this paper we prove a characterization theorem for the elements of the space $H^\prime_\mu$ of generalized functions defined by A.H. Zemanian.

KEY WORDS AND PHRASES. Generalized functions, Hankel transforms.

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1. INTRODUCTION.

The Hankel transformation defined by

$$h_\mu[f(x)](y) = \int_0^\infty (xy)^{1/2} J_\mu(xy)f(x)dx$$

where $J_\mu$ denotes the Bessel function of the first kind and order $\mu$, has been extensively studied in recent years.

A classical result concerning the Hankel transformation is the following inversion theorem (see [1]).

THEOREM 1. Let $f(x) \in L_1(0,\infty)$ be of bounded variation in a neighborhood of the point $x = x_0$. If $\mu > -\frac{1}{2}$ and $F(y) = h_\mu[f(x)](y)$, then

$$h^{-1}_\mu[F(y)](x_0) = \int_0^\infty F(y)(xy)^{1/2} J_\mu(xy)dy = \frac{1}{2} \{ f(x_0^+ + 0) + f(x_0^- - 0) \}.$$

Another well known result is the Parseval's equation (1) (see [1]).

THEOREM 2. Let $f(x)$ and $G(y)$ be elements of $L_1(0,\infty)$. If $F(y)$ and $g(x)$ are respectively the direct and inverse $\mu$-th order Hankel transforms of $f(x)$ and $G(y)$, then

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F(y)G(y)dy, \quad \text{for any } \mu > -\frac{1}{2}. \quad (1.1)$$
Other conditions under which Parseval's equation holds are given by P. Macaulay-Owen [2].

The $h$-transform has been extended to several spaces of generalized functions. Apparently, A.H. Zemanian [1] was the first to extend the Hankel transform. He introduced the space $H^\mu_\psi$ of testing functions consisting of all infinitely differentiable complex-valued functions $\psi$ defined on $I=(0,\infty)$ and such that

$$
\gamma_{m,n}^\mu_\psi (\psi) = \sup_{x \in I} x^m (x^{-\mu -1/2} \psi(x)) < \infty
$$

for every $m,n \in \mathbb{N}$. The Hankel transform is an automorphism onto $H^\mu_\psi$. For every $f \in H^\mu_\psi$ (the dual space of $H^\mu_\psi$), the generalized Hankel transformation $H^\mu_\psi f$ of $f$ was defined by the following generalization of Parseval's equation

$$
\langle H^\mu_\psi f, \psi \rangle = \langle f, H^\mu_\psi \psi \rangle, \text{ for every } \psi \in H^\mu_\psi.
$$

$h^\prime$ is an automorphism onto $H^\mu_\psi$.

Later, E.L. Koh and A.H. Zemanian [3] defined the generalized complex Hankel transformation. For a real number $\mu$ and a positive real number $\alpha$ the space $J^\mu_\alpha$ was defined as the space of testing functions $\psi$ which are smooth on $I$ and for which

$$
\gamma_k^\mu_\alpha (\psi) = \sup_{x \in I} e^{-\alpha x} x^{-\mu -1/2} S_\mu^k \psi(x) < \infty, \text{ for every } k \in \mathbb{N},
$$

where $S_\mu x = x^{-\mu -1/2} D_x^2 + D_x = x^{-\mu -1/2}$. For each complex number $y$ in the strip $
abla = \{ y \in C : \Re y < \alpha, y \in (-\infty,0) \}$, $J^\mu_\alpha$ contains the function $(xy)^{\mu -1/2} J^\mu_\alpha(xy)$. The $h^\mu$-transform is now defined on the dual space $J^\mu_\alpha$ as follows:

DEFINITION. Let $\mu$ be in the interval $-1/2 < \mu < \infty$. Then, for every $f \in J^\mu_\alpha$ and $y \in \nabla$,

$$
(h^\mu f)(y) = \langle f(x), (xy)^{\mu -1/2} J^\mu_\alpha(xy) \rangle.
$$

E.L. Koh [4] showed that a distribution $f \in J^\mu_\alpha$ can be written as a finite sum of derivatives of continuous function of exponential descent. More specifically, he established:

THOEREM 3 ([4]). Let $f$ be in $J^\mu_\alpha$. Then $f$ is equal to a finite sum

$$
\sum_{i=0}^k C_i (\frac{d}{dx})^i (e^{-\alpha x} x^{-\mu -1/2} - P_i(x)) F_i(x)
$$

where the $F_i(x)$ are continuous on $(0,\infty)$ and the $P_i(x)$ are polynomials of degree $k$.

Other Hankel type transformations have been also extended to certain spaces of generalized functions (see G. Altenburg [5], L.S. Dube and J.N. Pandey [6], J.M. Méndez [7], ...).
In this paper we prove a characterization theorem for the generalized functions in \( H'_\mu \). Our proof is analogous to the method employed in structure theorems for Schwartz distributions (see [8] and [4]).

In this paper a function \( \psi(x) \) will be called of rapid descent if \( x^m \partial^n \psi(x) \) tends to zero, as \( x \to +\infty \), for every \( m, n \in \mathbb{N} \).

2. The space \( H'_\mu \) of generalized functions. A characterization theorem.

A useful result due to A.H. Zemanian (see [1]) is the following

**Proposition 1.** Let \( f \) be in \( H'_\mu \). There exist a positive constant \( C \) and nonnegative integers \( r, k \) such that

\[
|<f, \psi>| \leq C \max\{\gamma_{m,n}(\psi); 0 < m < r, 0 < n < k\}, \text{ for every } \psi \in H'_\mu.
\]

We now present some new properties of the space \( H'_\mu \) of testing functions.

**Proposition 2.** Let \( \psi \) be in \( H'_\mu \). The function \( x^m \partial^n (x^{-\mu-1/2} \psi(x)) \) is

a) of rapid descent as \( x \to +\infty \), and

b) in \( L^1(0,\infty) \),

for every \( m, n \in \mathbb{N} \).

**Proof.** It is enough to take into account that

\[
|x^m \partial^n (x^{-\mu-1/2} \psi(x))| \leq C_{m,n} x^{-2}, \text{ for every } x \in I \text{ and } m, n \in \mathbb{N}, \text{ } C_{m,n} \text{ being a suitable positive constant.}
\]

The main result of this paper is the next.

**Theorem 4.** A functional \( f \) is in \( H'_\mu \) if and only if, there exist bounded measurable functions \( g_{m,n}(x) \) defined on \( I \), for \( m=0,1,\ldots,r \) and \( n=0,1,\ldots,k \), where \( r \) and \( k \) are nonnegative integers depending on \( f \), such that

\[
<f, \psi> = \int_{m,n} (-1)^n (x \partial^n (x^{-\mu-1/2} \psi(x))), \psi(x) > (2.1)
\]

for every \( \psi \in H'_\mu \).

**Proof.** Let \( f \) be in \( H'_\mu \). In view of Proposition 1, there exist a constant \( C > 0 \) and nonnegative integers \( r \) and \( k \) depending on \( f \) such that

\[
|<f, \psi>| \leq C \max\{\gamma_{m,n}(\psi); 0 < m < r, 0 < n < k\} \]

\[
=C \max\{\sup_{x \in I} |x^m \partial^n (x^{-\mu-1/2} \psi(x))|; 0 < m < r, 0 < n < k\},
\]

for every \( \psi \in H'_\mu \).
Since $x^{m-1/2}(x^{m-1/2} \psi(x))$ is of rapid descent as $x \to \infty$ (Proposition 2), we get

$$x^{m-1/2}(x^{m-1/2} \psi(x)) = \int_0^\infty D_t \{t^{m-1/2}(t^{m-1/2} \psi(t))\} dt$$

for every $\lambda \in H_\mu$, $m,n,e \in \mathbb{N}$.

Hence

$$\sup_{x \in \mathbb{R}} \left| x^{m-1/2}(x^{m-1/2} \psi(x)) \right| = \int_0^\infty \left| D_t \{t^{m-1/2}(t^{m-1/2} \psi(t))\} \right| dt$$

where $\| \cdot \|_{L^1(0,\infty)}$ denotes the norm on the space $L_1(0,\infty)$. Then we can write

$$\langle f, \psi \rangle = \text{Cmax} \left( \left\| D_t \{t^{m-1/2}(t^{m-1/2} \psi(t))\} \right\|_{L^1(0,\infty)} ; 0 \leq r, 0 \leq n \leq k \right)$$

for every $\psi \in H_\mu$.

We now define the injective map

$$F: H_\mu \rightarrow FH$$

$$\psi \rightarrow (D_t \{t^{m-1/2}(t^{m-1/2} \psi(x))\})_{m=0, \ldots, r}$$

$$n=0, \ldots, k$$

If $FH_\mu$ is endowed with the topology induced in it by the product space $A_r,k(0,\infty) = (L_1(0,\infty))^{(r+1)(k+1)}$, then

$$G: FH_\mu \rightarrow C$$

$$F \psi \rightarrow \langle f, \psi \rangle$$

is continuous linear mapping.

By application of the Hahn-Banach Theorem, $G$ can be extended to $A_r,k(0,\infty)$. Therefore, since $A_r,k(0,\infty)$ is isomorphic to $(L_1(0,\infty))^{(r+1)(k+1)}$ (see F. Treves [10]), there exist $(r+1)(k+1)$ bounded measurable functions, $g_{m,n}(m=0, \ldots, r; n=0, \ldots, k)$, such that:

$$G(F \psi) = \langle f, \psi \rangle = \sum_{m=0, n=0}^{r,k} g_{m,n}(x), D(x^{m-1/2}(x^{m-1/2} \psi(x)))$$

$$= \sum_{m=0, n=0}^{r,k} x^{m-1/2}(-x^{1/2}(x^{m-1/2} \psi(x))), \psi(x)$$

for every $\psi \in H_\mu$. 
On the other hand, if $f$ is defined by $(2)$ then $f \in H'$.

To see this, it is enough to prove that if $\{\psi_v\}$ is a sequence in $H_\mu$ such that $\psi_v \to 0$ as $v \to \infty$, then the sequence $\{x^{m-1}D_b(x)\psi_v(x)\}_{v \in \mathbb{N}}$ converges to zero as $v \to \infty$, in $L_1(0,\infty)$, for every $m, n \in \mathbb{N}$. This completes the proof of the theorem.

The Hankel-Schwartz transform defined by the pair

$$
F(y) = b^\mu f(x)(y) = \int_0^\infty x^{2\mu+1} b^\mu(xy)f(x)dx,
$$

$$
f(x) = b^\mu F(y)(x) = \int_0^\infty y^{2\mu+1} b^\mu(xy)F(y)dy,
$$

for $\mu > -\frac{1}{2}$, where $b^\mu(z) = z^{-\mu}J(\mu z)$ and $J$ denotes the Bessel function of the first kind and order $\mu$, was introduced by A.L. Schwartz [9], who established its inversion formula. This integral transformation has been extended by G. Altenburg [5] and J.M. Mendez [7] to the space $H^\mu_{1/2}$ of generalized functions ($H=H^0_{1/2}$ in their notation) following a procedure analogous to the one employed by A.H. Zemanian [1]. By setting $\mu = \frac{1}{2}$, we can deduce from Theorem 4 the next

**COROLLARY.** The functional $f$ is in $H'$ if and only if, there exist bounded measurable functions $g_{m,n}(x)$ defined on $I$, for $m=0, \ldots, r, n=0, \ldots, k$ where $r$ and $k$ are nonnegative integers depending on $f$, such that

$$
\langle f, \psi \rangle = \sum_{m=0}^{r} \sum_{n=0}^{k} \frac{(-1)^n}{x^m(-D)}g_{m,n}(x), \psi(x) \rangle, \psi \in H.
$$

**REFERENCES**


