A NOTE ON BAZILEVIĆ FUNCTIONS

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ABSTRACT. For $\alpha > 0$, let $B_1(\alpha)$ be the class of normalized analytic functions defined in the open unit disc $D$ satisfying $Re(f(z)/z)^{\alpha-1}f'(z) > 0$ for $z \in D$. The sharp lower bound for $Re(f(z)/z)^{\alpha}$ is obtained and the result is generalized to some iterated integral operators.

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INTRODUCTION

For $\alpha > 0$, let $B_1(\alpha)$ denote the class of Bazilević functions defined in the open unit disc $D = \{ z : |z| < 1 \}$ normalized so that $f(0) = 0$, $f'(0) = 1$ and such that for $z \in D$,

$$Re f'(z) \left( \frac{f(z)}{z} \right)^{\alpha-1} > 0. \quad (1)$$

This class of functions was studied first by Singh [4] and has been considered recently by several authors e.g. [2,3,5]. We note that $B_1(1) = R$, the class of functions whose derivative has positive real part.

For $f \in R$, Hallenbeck [1] showed that for $z = re^{i\theta} \in D$

$$Re \left( \frac{f(z)}{z} \right) \geq -1 + \frac{2}{r} \log(1+r) > -1 + 2\log 2;$$

with equality for the function $f_1(z) = -z + 2\log(1+z)$ and for $B_1(\alpha)$, the non-sharp estimate $Re(f(z)/z)^{\alpha} > 1/(1+2\alpha)$ was obtained in [3]. In this note, we give the sharp estimate for the lower bound of $Re(f(z)/z)^{\alpha}$ when $f \in B_1(\alpha)$ and extend the result to obtain sharp estimates for the real part of some iterated integral operators.

RESULTS

For $z \in D$ and $n = 1, 2, \ldots$, define

$$I_n(z) = \frac{1}{z} \int_0^z I_{n-1}(t) dt,$$

where $I_0(z) = (f(z)/z)^{\alpha}$.
Theorem. Let \( f \in B_1(\alpha) \) and \( z = re^{i\theta} \in D \). Then for \( n \geq 0 \),

\[
\text{Re } I_n(z) \geq \gamma_n(r) > \gamma_n(1),
\]

where

\[
0 < \gamma_n(r) = -1 + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+\alpha)} < 1.
\]

Equality occurs for the function \( f_\alpha \) defined by

\[
f_\alpha(z) = \left( \alpha \int_0^z t^\alpha - 1 \left( \frac{1-t}{1+t} \right) dt \right)^{1/\alpha}.
\]

We note that when \( n = 0 \),

\[
\text{Re} \left( \frac{f(z)}{z} \right)^\alpha \geq \frac{\alpha}{r^\alpha} \int_0^r \rho^{\alpha-1} \left( \frac{1-\rho}{1+\rho} \right) d\rho = -1 + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{(k-1+\alpha)},
\]

which reduces to \(-1 + (2/r) \log(1 + r)\) when \( \alpha = 1 \).

Proof: From (1) write

\[
f'(z) \left( \frac{f(z)}{z} \right)^{\alpha-1} = h(z),
\]

where \( h \in P \), i.e., \( h(0) = 1 \) and \( \text{Re } h(z) > 0 \) for \( z = re^{i\theta} \in D \).

Thus

\[
\text{Re} \left( \frac{f(z)}{z} \right)^\alpha = \alpha \text{Re} \left( \frac{1}{z^\alpha} \int_0^z t^\alpha - 1 h(t) dt \right).
\]

Write \( t = \rho e^{i\theta} \), so that

\[
\text{Re} \left( \frac{f(z)}{z} \right)^\alpha = \frac{\alpha}{r^\alpha} \int_0^r \rho^{\alpha-1} \text{Re } h(\rho e^{i\theta}) d\rho,
\]

\[
\geq \frac{\alpha}{r^\alpha} \int_0^r \rho^{\alpha-1} \left( \frac{1-\rho}{1+\rho} \right) d\rho,
\]

since \( h \in P \).

Hence

\[
\text{Re } I_0(z) = \text{Re} \left( \frac{f(z)}{z} \right)^\alpha \geq \frac{\alpha}{r^\alpha} \int_0^r \rho^{\alpha-1} \left( \frac{1-\rho}{1+\rho} \right) d\rho.
\]
Next

\[ \text{Re } I_{n+1}(z) = \text{Re } \frac{1}{z} \int_0^z I_n(t)dt, \]
\[ = \frac{1}{r} \int_0^r \text{Re } I_n(\rho e^{i\theta})d\rho, \]
\[ \geq \frac{1}{r} \int_0^r \left(-1 + 2\alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\rho^{k-1}}{k^n(k-1+\alpha)}\right)d\rho, \]
\[ = \gamma_{n+1}(r), \]

where the inequality follows by induction.

Now set

\[ \phi_n(r) = \alpha \sum_{k=1}^{\infty} \frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+\alpha)}. \]

This series is absolutely convergent for \( n \geq 0 \) and \( 0 < r < 1 \). Suitably rearranging pairs of terms in \( \phi_n(r) \) shows that \( \frac{1}{2} < \phi_n(r) < 1 \) and so \( 0 < \gamma_n(r) < 1 \).

Finally we note that since for \( n \geq 1 \)

\[ r\phi_n(r) = \int_0^r \phi_{n-1}(\rho)d\rho, \]

induction shows that \( \phi'_n(r) < 0 \) and so \( \gamma_n(r) \) decreases with \( r \) as \( r \to 1 \) for fixed \( n \) and increases to 1 as \( n \to \infty \) for fixed \( r \).

**REFERENCES**
