A SPECIAL PRIME DIVISOR OF THE SEQUENCE:
\[ Ah + B, A(h+1) + B, \ldots, A(h+k-1) + B \]

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1. INTRODUCTION. Schur showed \[1,2,3\] that for every pair of integers \( h, k \) where \( h \geq k \), at least one of the integers

\[ h + 1, \ h + 2, \ h + 3, \ldots, \ h + k, \]

is divisible by a prime \( p > k \).

Schur also showed \[1\] that for \( h > k > 2 \), one of the odd integers

\[ 2h + 1, \ 2(h+1) + 1, \ldots, \ 2(h+k-1) + 1 \]

is divisible by a prime \( p > 2k + 1 \). In this paper we generalize these two results by showing the following theorem.

THEOREM 1. Let \( A \) and \( B \) be two relatively prime positive integers. Then for \( h > k \) and sufficiently large \( k \), at least one of the integers

\[ Ah + B, \ A(h+1) + B, \ldots, A(h+k-1) + B \quad (1.1) \]

is divisible by a prime \( p \) such that

\[ p > Ak + B. \quad (1.2) \]

We need the following lemma.

LEMMA 1. Let \( \beta > 1 \) be given. Then for sufficiently large \( x \), there is always a prime \( p \) such that

\[ x < p \leq \beta x \quad \text{and} \quad p \equiv B \pmod{A}. \]

PROOF. Define the function \( \theta_A(x) \) by

\[ \theta_A(x) = \sum_{\substack{p \leq x \\atop p \equiv B \pmod{A}}} \log p, \]

where the sum is taken over all primes less than or equal to \( x \) and congruent to \( B \) modulo \( A \). Then the prime number theorem for an arithmetic progressions asserts that

\[ \theta_A(x) \sim \frac{x}{\varphi(A)}, \]
where \( \varphi(A) \) is the number of integers that are less than \( A \) and relatively prime to \( A \). Let \( \epsilon > 0 \) be given, then if \( x \) is sufficiently large we have

\[
(1 - \epsilon) \frac{\varphi(A)}{A} < \theta_A(x) < (1 + \epsilon) \frac{\varphi(A)}{A}.
\]

Thus

\[
\sum_{\substack{z < p \leq \beta x \\atop p \equiv B \pmod{A}}} \log p = \theta_A(\beta x) - \theta_A(x) > \frac{1}{\varphi(A)} [(1 - \epsilon)\beta x - (1 + \epsilon)x]
\]

\[
= \frac{\varphi(A)}{\varphi(A)} [\beta - 1 - \epsilon (\beta + 1)].
\]

If \( \epsilon \) is chosen so that \( 0 < \epsilon < \frac{\beta - 1}{\beta + 1} \), then

\[
\sum_{\substack{z < p \leq \beta x \\atop p \equiv B \pmod{A}}} \log p > 0.
\]

Thus if \( x \) is large, then there is at least one prime \( p \) such that \( x < p \leq \beta x \) and \( p \equiv B \pmod{A} \), and the lemma is proved.

**Proof of Theorem 1.** Suppose the theorem is false for a pair \((h, k)\), then the numbers

\[A(h + B), A(h + 1) + B, \ldots, A(h + k - 1) + B,\]

have only prime divisors which are less than or equal to \( Ak + B \). Consider

\[
G = \frac{(Ah + B)(A(h + 1) + B)\ldots(A(h + k - 1) + B)}{B(A + B)(2A + B)\ldots(Ak - A + B)} \quad (1.3)
\]

and let \( w_p \) be the integer exponent (positive, negative or zero) of \( p \) which appears in \( G \). Then by our assumption, every prime appearing in \( G \) is less than or equal to \( Ak + B \). Thus,

\[
G = \prod_{p \leq Ak + B} p^{w_p}. \quad (1.4)
\]

We claim that

\[
\begin{cases}
  w_p = 0 & \text{if } p \mid A \\
  w_p \leq \frac{\log(Ah + Bk)}{\log p} & \text{if } p \not\mid A
\end{cases}
\]

For if \( p \mid A \), then \( p \not\mid A j + B \) for any integer \( j \); otherwise we would have \( p \mid B \) and so \( p \) divides both \( A \) and \( B \). This is impossible, since \( A \) and \( B \) are relatively prime. Thus \( p \) does not divide any factor of either the numerator or the denominator of (1.3), hence \( w_p = 0 \).

Suppose now that \( p \not\mid A \); then it is easy to see that

\[
w_p = \sum_{1 < p^r \leq A(h + k - 1) + B} (U(p^r) - V(p^r)), \quad (1.6)
\]

where the sum is taken over all prime powers \( p^r \) between 1 and \( A(h + k - 1) + B \). \( U(p^r) \) is the number of factors in the numerator of (1.3) that are divisible by \( p^r \) and \( V(p^r) \) is the number of factors in the denominator of (1.3) that are divisible by \( p^r \).

Since \( Ax + B \equiv 0 \pmod{p^r} \) has only one solution for \( x \) modulo \( p^r \), \( Ax + B \) is divisible by \( p^r \) for only one value of \( x \) when \( x \) runs through \( p^r \) consecutive integers. Therefore,

\[
\left[ \frac{k}{p^r} \right] \leq U(p^r) \leq \left[ \frac{k}{p^r} \right] + 1, \quad \left[ \frac{k}{p^r} \right] \leq V(p^r) \leq \left[ \frac{k}{p^r} \right] + 1.
\]
Thus

\[-1 \leq U(p^r) - V(p^r) \leq 1.\]

This and (1.6) give

\[w_p \leq \sum_{p^r \leq A(h+k)} 1 \leq \frac{\log (Ah + Ak)}{\log p},\]

and the claim is proved. Thus

\[w_p \leq Ah + Ak, \quad \text{for all } p.\]

This and (1.4) give

\[G \leq \prod_{p \leq Ak + B} (Ah + Ak);\]

thus

\[G \leq (Ah + Ak)^{\pi(Ak + B)}. \tag{1.7}\]

On the other hand, by (1.3) we have

\[G = \prod_{j=1}^{k} \frac{A(h + j - 1) + B}{A(j - 1) + B} = \prod_{j=1}^{k} \frac{Ah + Aj - A + B}{Aj - A + B} = \prod_{j=1}^{k} \left(1 + \frac{Ah}{Aj - A + B}\right) \geq \prod_{j=1}^{k} \left(1 + \frac{Ah}{Aj}\right) \quad \text{(since } A > B) \geq \left(1 + \frac{h}{k}\right)^k,\]

or

\[G \geq \left(1 + \frac{h}{k}\right)^k. \tag{1.8}\]

Combining (1.7) and (1.8) yields

\[\left(1 + \frac{h}{k}\right)^k \leq (Ah + Ak)^{\pi(Ak + B)}.\]

Taking logarithms, we get

\[k \log \left(1 + \frac{h}{k}\right) \leq \pi(Ak + B) \log (Ah + Ak).\]

Writing \(\log (Ah + Ak) = \log Ak + \log \left(1 + \frac{h}{k}\right)\) gives

\[\{k - \pi(Ak + B)\} \log \left(1 + \frac{h}{k}\right) \leq \pi(Ak + B) \log Ak.\]

Dividing both sides of this inequality by \(Ak + B\), we get

\[\left\{\frac{k}{Ak + B} - \frac{\pi(Ak + B)}{Ak + B}\right\} \log \left(1 + \frac{h}{k}\right) \leq \frac{\pi(Ak + B) \log Ak}{Ak + B} \leq \frac{\pi(Ak + B) \log (Ak + B)}{Ak + B} \leq \frac{3}{2}.\]

Thus,

\[\left\{\frac{k}{Ak + B} - \frac{\pi(Ak + B)}{Ak + B}\right\} \log \left(1 + \frac{h}{k}\right) \leq \frac{3}{2}. \tag{1.9}\]
Consider two cases.

Case I. \( \frac{h}{k} \geq e^{2A} - 1 \)

Then \( \log \left( 1 + \frac{h}{k} \right) \geq 2A \). Using this in (1.9) we obtain

\[
\left\{ \frac{k}{Ak + B} - \frac{\pi(Ak + B)}{Ak + B} \right\}(2A) \leq \frac{3}{2}
\]

Letting \( k \to \infty \) in this inequality gives

\[
\frac{1}{A} \cdot 2A \leq \frac{3}{2},
\]

or

\[
2 \leq \frac{3}{2}.
\]

This provides a contradiction that proves the theorem in this case.

Case II. \( \frac{h}{k} < e^{2A} - 1 \)

Then

\[
\frac{Ah + Ak + B}{Ah} = 1 + \frac{B}{Ah}
\]

\[
> 1 + \frac{1}{e^{2A} - 1} + \frac{B}{Ah}
\]

or

\[
\frac{Ah + Ak + B}{Ah} \geq 1 + c,
\]

where \( c \) is a positive constant (depending only on \( A \)). Thus

\[
\frac{Ah + Ak + B}{Ah} > \beta, \quad \text{where } \beta = 1 + c > 1.
\]

By Lemma 1 if \( h \) is large (or \( k \) is large, since \( h > k \)), there exists a prime integer \( p \) such that \( p \equiv B \pmod{A} \) and

\[
Ah < p \leq \beta Ah < Ah + Ak + B.
\]

Thus

\[
Ah + B \leq p \leq Ah + Ak + B - A.
\]

Therefore one of the integers

\[
Ah + B, Ah + 1 + B, \ldots, Ah + k - 1 + B,
\]

is a prime \( p \). Since \( p \geq Ah + B \) and \( h > k \), then

\[
p > Ak + B,
\]

which is condition (1.2). This completes the proof of the theorem.

REFERENCES

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