A DISCRETE STOCHASTIC KOROVKIN THEOREM

GEORGE A. ANASTASSIOU
Department of Mathematical Sciences
Memphis State University
Memphis, Tennessee 38152 U.S.A.

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ABSTRACT. In this article we give a sufficient condition for the pointwise -- in
the first mean Korovkin property on $B_0(P)$, the space of stochastic processes with
real state space and countable index set $P$ and bounded first moments.

KEY WORDS AND PHRASES. Positive linear operator, stochastic processes, pointwise
- in the first mean convergence.

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Secondary 60G99.

1. INTRODUCTION.

Let $(\Omega_0, A_0, \tau)$ be a probability space and let $P$ denote a fixed countable set.
Consider stochastic processes $X$ with real state space and the expectation operator
$E(X)(t) = \int X(t, \omega) \tau(\omega) d\omega$, $t \in P$. Define $B_0(P) = \{X: \sup_{t \in P} E[X](t) < \infty\}$. Let $T_n: B_0(P) \rightarrow B_0(P)$ be any sequence of positive linear operators such that $E(T_n) = T_n E$, all
$n = 1, 2, \ldots$. In Theorem 1, under Korovkin type assumptions, we give a sufficient
condition such that for each $X \in B_0(P),$

$$
\lim_{n \to \infty} E[(T_n X)(t, \omega) - X(t, \omega)] = 0, \text{for each } t \in P.
$$

In [3], see Theorem 3.2, was treated the continuous case, that is, when $P$ is an un-
countable compact space. There the sufficient condition is similar to ours, however,
it is produced under the additional assumption that $T_n$ is a stochastically simple
operator.

Our result has as follows:

THEOREM 1. Let $(\Omega_0, A_0, \tau)$ be a probability space and $P = \{t_1, \ldots, t_j, \ldots\}$ be a
countable set of cardinality $\geq 2$. Consider the space of stochastic processes with
real state space

$$
B_0(P) = \{X: \sup_{t \in P} |X(t, \omega)| \tau(\omega) < \infty\}
$$

and the space

$$
B(P) = \{f: P \rightarrow \mathbb{R} \mid \|f\|_{\infty} < \infty\},
$$

where

$$
\|f\|_{\infty} = \sup_{t \in P} |f(t)|; B(P) \subset B_0(P).
$$

Let $T_n: B_0(P) \rightarrow B_0(P)$ be a sequence of positive linear operators that are $E$-commu-
tative, i.e.

$$
(E(T_n X))(t, \omega) = (T_n E(X))(t, \omega), \text{ for all } (t, \omega) \in P \times \Omega_0,
$$

and

$$
\lim_{n \to \infty} E[(T_n X)(t, \omega) - X(t, \omega)] = 0, \text{for each } t \in P.
$$
where
\[(EX)(t) := E(X(t,\omega)) := \int_{\Omega} X(t,\omega) \tau(d\omega)\]
is the expectation.

Also assume that \((T_n 1)(t,\omega) = 1\), for all \((t,\omega) \in \mathbb{P} \times \Omega\). For
\[\{X_1(t,\omega),\ldots,X_k(t,\omega)\} \subset B_0(\mathbb{P})\]
assume that
\[\lim_{n \to \infty} E [(T_n X_i)(t_j,\omega) - X_i(t_j,\omega)] = 0,\]
for all \(t_j \in \mathbb{P}\) and all \(i = 1,\ldots,k\). (i.e.
\[\lim_{n \to \infty} [(T_n(EX_i))(t_j) - (EX_i)(t_j)] = 0,\]
for all \(t_j \in \mathbb{P}\) and \(i = 1,\ldots,k\).

In order that
\[\lim_{n \to \infty} E [(T_n X)(t_j,\omega) - X(t_j,\omega)] = 0,\]
for all \(t_j \in \mathbb{P}\) and all \(X \in B_0(\mathbb{P})\), it is enough to assume that each \(t_j \in \mathbb{P}\) there
are real constants \(\beta_1,\ldots,\beta_k\) such that
\[\sum_{i=1}^{k} \beta_i E[X_i(t,\omega) - X_i(t_j,\omega)] \geq 1, \text{ for all } t \in \mathbb{P} - \{t_j\}.

PROOF. If there exists \(X \in B_0(\mathbb{P})\) and \(t_j \in \mathbb{P}\) such that
\[E [(T_n X)(t_j,\omega) - X(t_j,\omega)] \neq 0,\]
then there exist a subsequence \(T_n^{(j)}\) and an \(\varepsilon > 0\) such that
\[|(E(T_n X))(t_j) - (EX)(t_j)| > \varepsilon, \text{ for all } n \geq 1.\]

By E-commutativity of \(T_n^{(j)}\) we get
\[|(T_n^{(j)}(EX))(t_j) - (EX)(t_j)| > \varepsilon, \text{ for all } n \geq 1.\]

Let \(\mu\) be a positive finite measure on \(\mathbb{P}\) with \(\mu(t) > 0\), for all \(t \in \mathbb{P}\). Here
\(B(\mathbb{P}) \subset L_p(\mathbb{P},\mu), 1 \leq p < \infty.\)

Let \(f \in B(\mathbb{P})\), then \(E(f) = f\). Hence \(T_n(f) = T_n(Ef) = ET_n(f)\) and \(T_n(f) \in B(\mathbb{P})\),
i.e. \(T_n\) maps \(B(\mathbb{P})\) into itself. Because each positive linear functional \(T_n(\cdot,t_j)\)
on \(B(\mathbb{P})\) is bounded, by Riesz representation theorem, for the specific \(j = j_0\),
there exists \(g_{t_j_0}^1 \in L_q(\mathbb{P},\mu)\) where \(\frac{1}{p} + \frac{1}{q} = 1\) such that
\[(T_n(f))(t_j_0) = \int_{\mathbb{P}} f(t) g_{t_j_0}^1(t) \mu(dt), \text{ for all } f \in B(\mathbb{P}).\]

By \(T_n(1) = 1\) and the positivity of \(T_n(\cdot,t_j)\) one obtains
\[\int_{\mathbb{P}} g_{t_j_0}^1(t) \mu(dt) = 1 \quad \text{and} \quad g_{t_j_0}^1(t) \geq 0, \text{ for all } t \in \mathbb{P}.\]

Since \(EX \in B(\mathbb{P})\), we have
\[(T_n^{(j_0)}(EX))(t_j) = \int_{\mathbb{P}} (EX)(t) \cdot g_{t_j_0}^1(t) \cdot \mu(dt).\]
Thus
\[ \varepsilon < |(T_n(\lambda \cdot (EX))(t_j) - (EX)(t_j)| = \left| \int P (EX(t) \cdot g_{t_j,\lambda_n} (t) \cdot \mu(dt) - \int P (EX(t_j) \cdot g_{t_j,\lambda_n} (t) \cdot \mu(dt) \right| \]
\[ = \left| \int_{P - \{t_j\}} [(EX(t) - (EX)(t_j)] \cdot g_{t_j,\lambda_n} (t) \cdot \mu(dt) \right| \]
\[ \leq ||EX - (EX)(t_j)||_\infty \cdot \left( \int_{P - \{t_j\}} g_{t_j,\lambda_n} (t) \mu(dt) \right), \]
so that
\[ \int_{P - \{t_j\}} g_{t_j,\lambda_n} (t) \mu(dt) \geq \frac{\varepsilon}{||EX - (EX)(t_j)||_\infty} =: \delta > 0, \text{ for all } n > 1. \]

There cannot be real constants \( \beta_1, \ldots, \beta_k \) with
\[ \sum_{i=1}^k \beta_i E[X_i(t, \omega) - X_i(t_j, \omega)] \geq 1, \text{ for all } t \in P - \{t_j\}. \]

Since, otherwise, we would have
\[ \sum_{i=1}^k \beta_i \int_{P - \{t_j\}} [(EX_i(t) - (EX_i)(t_j)] \cdot g_{t_j,\lambda_n} (t) \cdot \mu(dt) \]
\[ \geq \int_{P - \{t_j\}} g_{t_j,\lambda_n} (t) \cdot \mu(dt) > \delta. \]

(Note that
\[ (T_n(\lambda \cdot (EX_i))(t_j) = \int P (EX_i(t) \cdot g_{t_j,\lambda_n} (t) \cdot \mu(dt), \text{ for all } i = 1, \ldots, k.) \]

However, from the assumptions of the theorem, we have
\[ \lim_{n \to \infty} (T_n(\lambda \cdot (EX_i))(t_j) = (EX_i)(t_j), \text{ all } i = 1, \ldots, k. \]

Hence
\[ 0 = \lim_{n \to \infty} \left( \sum_{i=1}^k \beta_i [(T_n(\lambda \cdot (EX_i))(t_j) - (EX_i)(t_j)] > \delta. \]

Thus \( \delta < 0 \), contradicting \( \delta > 0 \). □

To show that the assumptions of Theorem 1 are not empty and they are powerful, we present

EXAMPLE 2. (i) Consider the probability space \([-a,a], B, \lambda \cdot \frac{\lambda}{2a}\), where \( a > 0 \),
\( B \) the Borel \( \sigma \)-algebra on \([-a,a]\), \( \lambda \) the Lebesgue measure on \([-a,a]\). Since
\( \frac{\lambda}{2a}([-a,a]) = 1, \frac{\lambda}{2a} \) is a probability measure on \([-a,a]\). Let also \( P = \{1, \pm 2, \ldots, \pm T\} \) be a finite set of integers. That is here \( \omega \in \Omega = [-a,a] \) and \( t \in P \).
Consider the sequence of operators

\[ T_n : B_0^0(P) \to B_0^0(P) \]

such that

\[ (T_nX)(t,\omega) = X(t,\omega)(1 - e^{-n|t|}) + X(-t,\omega)e^{-n|t|}, \]

for all \( n \geq 1 \).

If \( X \geq 0 \) then \( T_nX \geq 0 \), that is \( T_n \) is a positive operator, furthermore \( T_n(1) = 1 \), for all \( n \geq 1 \). It is obvious that \( T_n \) is linear.

Observe that

\[ (E(T_nX))(t,\omega) = (EX)(t)\cdot(1 - e^{-n|t|}) + (EX)(-t)e^{-n|t|} \]

i.e., \( ET_n = T_nE \), that is \( T_n \) is \( E \)-commutative for all \( n \geq 1 \). Therefore \( T_n \) fulfills the assumption of Theorem 1.

From

\[ (E(T_nX))(t) = (EX)(t)\cdot(1 - e^{-n|t|}) + (EX)(-t)e^{-n|t|}, \]

it is clear that

\[ \lim_{n \to \infty} E[(T_nX)(t,\omega) - X(t,\omega)] = 0, \]

for all \( t \in P \) and all \( X \in B_0^0(P) \). Thus \( T_n \) fulfills the conclusion of Theorem 1.

(ii) Continuing in the setting of part (i): Let \( X_1(t,\omega) = 1 \), \( X_2(t,\omega) = 2t|\omega|/a \)

and \( X_3(t,\omega) = 3t^2\omega^2/a^2 \). Then \( (EX_1)(t) = 1 \), \( (EX_2)(t) = t \) and \( (EX_3)(t) = t^2 \). It is obvious that \( X_1, X_2, X_3 \in B_0^0(P) \). We would like to find \( \beta_1, \beta_2, \beta_3 \) such that

\[ \sum_{i=1}^{3} \beta_i[(EX_i)(t) - (EX_i)(t_j)] \geq 1, \]

for all \( t \in P - \{t_j\} \).

For that we can pick \( \beta_1 \) an arbitrary real number, \( \beta_2 = -2t_j \) and \( \beta_3 = 1 \). We have

\[ \beta_1(1 - 1) + (-2t_j)(t - t_j) + (t^2 - t_j^2) = (t - t_j)^2 \geq 1, \]

for all \( t \in P - \{t_j\} \). Hence \( X_i, i = 1,2,3 \) fulfill the sufficient condition of Theorem 1.

Trivially \( T_nX_i = X_i \), giving us \( ET_nX_i = EX_i \), for \( i = 1,3 \). And

\[ (T_nX_2)(t,\omega) = X_2(t,\omega)(1 - e^{-n|t|}) + X_2(-t,\omega)e^{-n|t|}, \]

implying

\[ (E(T_nX_2))(t) = t(1 - 2e^{-n|t|}). \]

Clearly

\[ \lim_{n \to \infty} (E(T_nX_2))(t) = (EX_2)(t). \]

We have seen how \( X_i, i = 1,2,3 \) fulfill the assumptions of Theorem 1.

REFERENCES

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