ON A GENERALIZATION OF \( u \)-MEANS

FRANCOIS DUBEAU

Département de mathématiques
Collège militaire royal de Saint-Jean
Saint-Jean-sur-Richelieu, Québec, Canada, JOJ 1RO

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ABSTRACT. In this paper we present an extension of Bauer's work about \( u \)-means. We consider a kind of composition of an admissible function \( u(x) \) (described by Bauer) and of a compatible function \( \phi(x) \). This construction allows us to define \( (u, \phi) \)-means. When \( \phi(x) = x \), the \( (u, \phi) \)-means are the \( u \)-means introduced by Bauer. The arithmetic, geometric and harmonic means are special cases.

KEY WORDS AND PHRASES. Means, \( u \)-means, generalized \( u \)-means.


1. INTRODUCTION.

In [2] Bauer introduced a class of admissible functions. To each function \( u(x) \) in this class it was possible to associate a \( u \)-mean. The arithmetic and geometric means were special cases of \( u \)-means but not the harmonic mean.

In this paper we introduce a class of monotone compatible functions. We consider a kind of composition of an admissible function \( u(x) \) and a monotone function \( \phi(x) \) compatible with respect to \( u(x) \) which permits the definition of \( (u, \phi) \)-means. When \( \phi(x) = x \) the \( (u, \phi) \)-means are the \( u \)-means of Bauer. The arithmetic, geometric and harmonic means are special cases.

2. CONTRACTIVE INTERVAL.

In this paper we consider intervals \( I \subset [0, +\infty[ \) of the following type

(i) \( ]0, +\infty[ \),
(ii) \( ]0, \alpha] \) or \( ]0, \alpha[ \) for \( 0 < \alpha \leq 1 \),
(iii) \( ]\beta, +\infty[ \) or \( ]\beta, +\infty[ \) for \( 1 \leq \beta < +\infty \).

Any interval \( I \) of this type is said to be \textit{contractive} because for any \( n \in N = \{1, 2, 3, \cdots\} \) and \( x \in I \) we have \( x^n \in I \), or equivalently, for any \( x, y \in I \) we have \( xy \in I \) (see [2]).

3. CLASSES OF FUNCTIONS.

The first class of functions we consider is the class of admissible functions introduced by Bauer [2].

A strictly positive continuous function \( u(x) \) defined on a contractive interval \( I_u \) is said \textit{admissible} (of type (A) or (B)) if it satisfies one of the following conditions:

(A) \( x \rightarrow u(x) \) is decreasing,
(B) \( x \rightarrow u(x)/x \) is strictly increasing.

\textbf{EXAMPLE 1.} \( u(x) = x^p \) for \( p \leq 0 \) or \( p > 1 \) are admissible functions on \( I_u \). The function \( u(x) = \sqrt{1 - x^2} \) is admissible on \( I_u = ]0, 1[ \). The function \( u(x) = e^{x^2} - 1 \) is admissible on \( [1, +\infty[ \).

To extend the work of Bauer we introduce the following class of functions. A strictly positive
strictly monotone continuous function \( \phi(x) \) defined on a contractive interval \( I_\phi \) is said compatible if it satisfies the following condition:

\[
x \rightarrow \phi(\alpha x)/\phi(x)
\]
is monotone (as \( \phi(x) \)) for any \( \alpha \in I_\phi \).

Let us consider the following examples.

**EXAMPLE 2.** \( \phi(x) = x^p \) is strictly increasing for \( p > 0 \) and strictly decreasing for \( p < 0 \). Also \( \phi(\alpha x)/\phi(x) = \alpha^p \), a constant for any fixed \( \alpha \in I_\phi \). Note that in this case \( \phi(x^n) = [\phi(x)]^n \).

**EXAMPLE 3.** \( \phi(x) = e^x \) and \( I_\phi = [1, +\infty[ \). The function \( \phi(x) \) is a strictly increasing continuous function such that \( \phi(\alpha x)/\phi(x) = e^{(\alpha - 1)x} \) which is an increasing function for any fixed \( \alpha \in I_\phi \).

**EXAMPLE 4.** \( \phi(x) = e^{-x} \) and \( I_\phi = [0, 1] \). The function \( \phi(x) \) is a strictly increasing function on \( I_\phi \) such that \( \phi(\alpha x)/\phi(x) = e^{-(1-\alpha)x} \) which is an increasing function for any fixed \( \alpha \in I_\phi \).

The following preliminary results will be useful in the next section.

**LEMMA 1.** Let \( \phi(x) \) be a compatible function. Then

(i) \( \phi(\alpha x)/\phi(x) \) is strictly monotone (as \( \phi(x) \)) for any integer \( n \in \mathbb{N} \),
(ii) \( \phi(\alpha x)/\phi(x) \) is monotone (as \( \phi(x) \)) for any \( n \in \mathbb{N} \) and any fixed \( \alpha \in I_\phi \).

**PROOF.** Let us assume first that \( \phi(x) \) is strictly increasing. To prove (i) consider \( x < y \), then

\[
x^n < x^{n-1}y < x^{n-1}y^n < y^n.
\]
Hence \( \phi(x^n)/\phi(x) < \phi(x^{n-1})/\phi(x) \leq \phi(y^n)/\phi(y) \) because \( \phi(x) \) is strictly increasing and compatible. To prove (ii) replace \( \alpha \) by \( \alpha^n \) in the definition. The proof is almost the same when \( \phi(x) \) is strictly decreasing.

**LEMMA 2.** Let \( u(x) \) be an admissible function and \( \phi(x) \) be a compatible function. If \( \phi(I_\phi) \subset I_u \), then for any \( n \in \mathbb{N} \) the function \( x \rightarrow \psi_{n+1}(x) = u_\phi(x^n)/\phi(x) \) is strictly monotone (here \( u_\phi(x) = u(\phi(x)) \)). The different cases are summarized in the following table:

<table>
<thead>
<tr>
<th>type of ( u(x) )</th>
<th>( \phi(x) ) strictly monotone</th>
<th>( \psi_{n+1}(x) ) strictly monotone</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) increasing</td>
<td>increasing</td>
<td>decreasing</td>
</tr>
<tr>
<td>(A) decreasing</td>
<td>decreasing</td>
<td>increasing</td>
</tr>
<tr>
<td>(B) increasing</td>
<td>increasing</td>
<td>increasing</td>
</tr>
<tr>
<td>(B) decreasing</td>
<td>decreasing</td>
<td>decreasing</td>
</tr>
</tbody>
</table>

**PROOF.** Let us assume that \( \phi(x) \) is strictly increasing (decreasing). If \( u(x) \) is of type (A) then \( u_\phi(x) \) is decreasing (increasing). Also \( 1/\phi(x) \) is strictly decreasing (increasing). It follows that \( u_\phi(x^n)/\phi(x) \) is strictly decreasing (increasing). If \( u(x) \) is of type (B) then \( u_\phi(x)/\phi(x) \) is strictly increasing (decreasing). From Lemma 1, \( \phi(x^n)/\phi(x) \) is strictly increasing (decreasing). The result follows from \( u_\phi(x^n)/\phi(x) = u_\phi(x^n)/\phi(x^n) \) \( = [\phi(x^n)/\phi(x^n)] \phi(x^n) \).

4. \((u, \phi)\)-MEANS. Let \( u(x) \) be an admissible function and \( \phi(x) \) be a compatible function such that \( \phi(I_\phi) \subset I_u \). Let \( n \geq 2 \) and choose any \( \vec{a} = (a_1, a_2, \ldots, a_n) \in I_\phi^n = I_\phi \times \cdots \times I_\phi \). We consider

\[
S_{(u, \phi)}(\vec{a}) = \frac{\sum_{i=1}^n u_\phi(\pi_1(\vec{a}))}{\sum_{i=1}^n \phi(a_i)}
\]

where

\[
\pi_1(\vec{a}) = \prod_{i=1}^n a_j = \prod_{j=1}^n a_j = \prod_{j=1}^n a_j / a_i.
\]

Using now the continuity of the functions and the strict monotonicity of \( \psi_n(x) = u_\phi(x^n-1)/\phi(x) \), we can prove the following result (which is a generalization of Theorem 2.1 of Bauer).
THEOREM 3. Let \( u(x) \) be an admissible function defined on \( I_u \) and \( \phi(x) \) a compatible function defined on \( I_\phi \) such that \( \phi(I_\phi) \subset I_u \). Let \( n \geq 2 \) and \( \overline{a} = (a_1, \ldots, a_n) \in I_\phi^n \). Then the equation

\[
\psi_n(x) = S_{(u, \phi)}(\overline{a})
\]

(4.1)

has exactly one solution in \( I_\phi \). It lies in the interval

\[
[\alpha, \beta] \text{ if } \alpha = \min \{a_1, \ldots, a_n\} < \max \{a_1, \ldots, a_n\} = \beta
\]

and is equal to \( \alpha \) if \( \alpha = \beta \).

NOTE. With the assumptions made on \( \phi(x) \) and the preceding two lemmas, the proof of this theorem is almost identical to the proof of Theorem 2.1 of Bauer.

PROOF. If \( \alpha = \beta \) the result follows from Lemma 2. Let us assume that \( \alpha \neq \beta \) and let us consider the following two cases:

(i) \( u(x) \) is of type (A) and \( \phi(x) \) is strictly decreasing. In this case \( uo\phi(x) \) is increasing. For any \( i = 1, \ldots, n \) we have \( a_{n-1}^\alpha \leq \phi \pi_1(\overline{a}) \leq \beta_{n-1}^\alpha \) and it follows that \( uo\phi(\alpha_{n-1}) \leq uo\phi(\pi_1(\overline{a})) \leq uo\phi(\beta_{n-1}) \). Also \( 1/\phi(x) \) is strictly increasing, then we have \( \phi(a_i)/\phi(\alpha) \leq 1 = \phi(a_i)/\phi(\beta) \) with strict inequality for at least one \( i \) (not necessarily the same \( i \) for both inequalities). It follows that

\[
\phi(a_i) \psi_n(\alpha) \leq uo\phi(\pi_1(\overline{a})) \leq \phi(a_i) \psi_n(\beta)
\]

(4.2)

for \( i = 1, \ldots, n \).

(ii) \( u(x) \) is of type (B) and \( \phi(x) \) is strictly increasing. In this case \( uo\phi(x)/\phi(x) \) is strictly increasing and we have

\[
\frac{uo\phi(\alpha_{n-1})}{\phi(\alpha_{n-1})} \leq \frac{uo\phi(\pi_1(\overline{a}))}{\phi(\pi_1(\overline{a}))} \leq \frac{uo\phi(\beta_{n-1})}{\phi(\beta_{n-1})}
\]

and again with strict inequality for at least one \( i \). We also have

\[
\phi(\alpha_{n-2} a_{i+1}) \leq \phi(\pi_1(\overline{a})) \leq \phi(\beta_{n-2} a_{i+1})
\]

for \( i = 1, \ldots, n \) (where \( a_{n+1} = a_1 \)). From Lemma 1 we have

\[
\phi(\alpha_{n-2})/\phi(\alpha) \leq \phi(\alpha_{n-2} a_{i+1})/\phi(a_{i+1}) \quad \text{and} \quad \phi(\beta_{n-2} a_{i+1})/\phi(a_{i+1}) \leq \phi(\beta_{n-1})/\phi(\beta).
\]

It follows that

\[
\phi(a_{i+1}) \psi_n(\alpha) \leq uo\phi(\pi_1(\overline{a})) \leq \phi(a_{i+1}) \psi_n(\beta).
\]

(4.3)

for \( i = 1, \ldots, n \).

By adding up (4.2) or (4.3) for \( i = 1, \ldots, n \) it follows that \( \psi_n(\alpha) < S_{(u, \phi)}(\overline{a}) < \psi_n(\beta) \) and the result follows from the continuity and the strict monotonicity of \( \psi_n(x) \).

For the other cases we obtain reverse inequalities and the result follows again.

Under the assumptions of Theorem 3, the \( (u, \phi) \)-mean of the \( n \) numbers \( a_1, \ldots, a_n \) taken in \( I_\phi \) will be the unique solution of (4.1) and will be denoted \( M_{(u, \phi)}(\overline{a}) \). If \( n = 1 \) we put \( M_{(u, \phi)}(a_1) = a_1 \).

REMARK 1. The \( u \)-means introduced by Bauer, denoted \( M_u(\overline{a}) \), are obtained when \( \phi(x) \) is the identity function \( id(x) \), i.e. \( \phi(x) = id(x) = x \) for any \( x \in I_\phi \), and we have \( M_{(u, id)}(\overline{a}) = M_u(\overline{a}) \).

REMARK 2. For \( u(x) = 1 \) we have
\[
\phi(M_{(1, \phi)}(\bar{a})) = A(\phi(\bar{a}))
\]

(4.4)

where \( \phi(\bar{a}) \) denotes the vector \((\phi(a_1), \cdots, \phi(a_n))\) and \(A(\bar{v})\) is the arithmetic mean of the \(n\) components of the vector \(\bar{v} = (v_1, \cdots, v_n)\).

**EXAMPLE 5.** Consider \(u(x) = 1\). For \(\phi(x) = \text{id}(x) = x\) we obtain

\[M_{(1, \text{id})}(\bar{a}) = A(\bar{a})\]

and for \(\phi(x) = 1/x = (1/\text{id}(x))\) since

\[H(a_1, \cdots, a_n) = A(a_1^{-1}, \cdots, a_n^{-1})\]

it follows from (4.4) that

\[M_{(1, 1/\text{id})}(\bar{a}) = H(\bar{a})\]

where \(H(\bar{a})\) denotes the harmonic mean of the \(n\) components of \(\bar{a}\). Let us note that it is not possible to obtain the harmonic mean as a u-means (see [1], [2]).

**REMARK 3.** More generally, if the function \(\phi(x)\) is such that

\[\phi(a_1, \cdots, a_n) = \prod_{i=1}^{n} \phi(a_i)\]

then we have

\[S(u, \phi)(\bar{a}) = S(u, \text{id})(\phi(\bar{a}))\]

and it follows that

\[\phi(M_{u, \phi}(\bar{a})) = M_{u, \text{id}}(\phi(a)) = M_u(\phi(\bar{a})).\]

(4.5)

**EXAMPLE 6.** For \(u(x) = 1/x = (1/\text{id}) (x)\), if \(\phi(x) = \text{id}(x)\) we obtain

\[M_{(1/\text{id}, \text{id})}(\bar{a}) = G(\bar{a})\]

where \(G(\bar{a})\) is the geometric mean of the \(n\) components of \(\bar{a}\). If \(\phi(x) = 1/x = (1/\text{id}) (x)\) it follows from (4.5) that

\[M_{(1/\text{id}, 1/\text{id})}(\bar{a}) = G(\bar{a})\]

because \(G(a_1^{-1}, \cdots, a_n^{-1}) = G(a_1, \cdots, a_n)^{-1}\).

**EXAMPLE 7.** More generally if \(u_p(x) = x^p\) (for \(p \leq 0\) or \(p > 1\)) and \(\phi(x) = x = \text{id}(x)\) we have

\[M_{(u_p, \text{id})}(\bar{a}) = \left[ \frac{G_{u_p}(a_1, \cdots, a_n)}{A(a_1, \cdots, a_n) H(a_1^p, \cdots, a_n^p)} \right]^{1 - p - 1}
\]

(see [2]). For \(\phi(x) = 1/x = (1/\text{id}) (x)\) we have

\[M_{(u_p, 1/\text{id})}(\bar{a}) = M_{(u_p, \text{id})}(1 - 1/\text{id}(\bar{a}))^{-1}
\]

\[= \left[ \frac{G_{u_p}(a_1, \cdots, a_n)}{A(a_1^p, \cdots, a_n^p) H(a_1, \cdots, a_n)} \right]^{1 - p - 1}.\]
EXAMPLE 8. Consider $\phi(x) = e^x$ on $I_\phi = [1, +\infty[$. If $u(x) = 1$ then

$$M_{(1, \phi)}(\alpha) = \ln \left( \frac{1}{n} \sum_{i=1}^{n} e^{\alpha_i} \right) = \ln (A(\phi(a_1), \cdots, \phi(a_n))).$$

More generally, if $u_p(x) = x^p (p \leq 0 \text{ or } > 1)$ we have that $M_{(u_p, \phi)}(\alpha)$ is the unique positive solution, not smaller than 1, of the polynomial equation

$$pM^{n-1} - M = \ln \left[ \frac{\sum_{i=1}^{n} e^{\alpha_i}}{\sum_{i=1}^{n} \exp (\alpha_i)} \right]$$

5. APPLICATIONS TO INEQUALITIES. In [2] Bauer presented inequalities between $u$-means (or $(u, id)$-means) and the arithmetic mean. Using the relation (4.5) it is possible to obtain similar inequalities for the harmonic mean. In fact we have the following results.

THEOREM 4. Let $u(x)$ be a convex admissible function of type (A) defined on the interval $I_u \supset \phi(I_\phi)$. For every choice of finitely many numbers $a_1, \cdots, a_n \in I_u$, if

(i) $\phi(x) = id(x)$ then $M_{(u, id)}(a_1, \cdots, a_n) \leq A(a_1, \cdots, a_n)$,

(ii) $\phi(x) = (1/id)(x)$ then $M_{(u, 1/id)}(a_1, \cdots, a_n) \geq H(a_1, \cdots, a_n)$.

Moreover if $u(x)$ is strictly convex then strict inequalities hold provided that $a_1, \cdots, a_n$ are not all equal.

THEOREM 5. Let $u(x)$ be a concave admissible function of type (B) defined on $I_u \supset \phi(I_\phi)$. For every choice of finitely many numbers $a_1, \cdots, a_n \in I_u$, if

(i) $\phi(x) = id(x)$ then $M_{(u, id)}(a_1, \cdots, a_n) \leq A(a_1, \cdots, a_n)$,

(ii) $\phi(x) = (1/id)(x)$ then $M_{(u, 1/id)}(a_1, \cdots, a_n) \geq H(a_1, \cdots, a_n)$.

Strict inequalities hold for $n \geq 3$ provided that $a_1, \cdots, a_n$ are not all equal.

The parts (i) of these two theorems are the results presented by Bauer in [2] because $M_u(\alpha) = M_{(u, id)}(\alpha)$. To prove the parts (ii) we only have to consider the relation (4.5) to obtain

$$M_{(u, 1/id)}(a_1, \cdots, a_n) = 1/M_u(a_1^{-1}, \cdots, a_n^{-1}).$$

Then, using the parts (i) we have

$$M_u(a_1^{-1}, \cdots, a_n^{-1}) \leq A(a_1^{-1}, \cdots, a_n^{-1})$$

but $A(a_1^{-1}, \cdots, a_n^{-1}) = H(a_1, \cdots, a_n)^{-1}$ and the results follow.

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REFERENCES
