LUCAS NUMBERS OF THE FORM Px^2 , WHERE P IS PRIME

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ABSTRACT. Let L_n denote the n^{th} Lucas number, where n is a natural number. Using elementary techniques, we find all solutions of the equation: $L_n = px^2$ where p is prime and $p < 1000$.

KEY WORDS AND PHRASES. Lucas number

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1. INTRODUCTION

Let n denote a natural number. Let L_n denote the n^{th} Lucas number, that is, $L_1=1$, $L_2=3$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 3$. In [1], J.H.E. Cohn found all Lucas numbers which are square or twice a square. As a result of a later paper of Cohn [2], it is known that for each integer $c \geq 3$, there is at most one Lucas number of the form cx^2 . Using [3], Definition 2, and (9) below, we see that there are 111 primes, p , such that (i) $2 < p < 1000$, and (ii) there exists n such that $p | L_n$. In this paper, we find all solutions of the equation:

$$L_n = px^2 \quad (*)$$

where the prime p satisfies conditions (i) and (ii) above. We find that only 8 such values of p yield solutions of (*). The results are summarized in Table 3 on the last page. The larger problem of finding all solutions to (*) appears more difficult; its solution would yield all Lucas numbers which are prime.

2. PRELIMINARIES

Let n denote a natural number. Let p denote a prime, not necessarily satisfying conditions (i) and (ii) above.

Definition 1 Let F_n denote the n^{th} Fibonacci number, that is, $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Definition 2 Let $z(n) = \min \{k: k \geq 1 \text{ and } n | F_k\}$.

Definition 3 Let $y(n) = \frac{1}{2}z(n)$ if $2 | z(n)$.

For each integer $c \geq 3$, the equation $L_n = cx^2$ has at most one solution. (1)

If $L_n = q$ is prime, then $y(q) = n$ and $y(q^2) = qn$. (2)

$$(L_n, L_{3n}/L_n) = \begin{cases} 3 & \text{if } n \equiv 2 \pmod{4} \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

$$L_n = x^2 \text{ iff } n=1 \text{ or } 3 \quad (4)$$

$$L_{3n} = L_n(L_n^2 - 3(-1)^n) \quad (5)$$

$$\text{If } m \text{ is odd and } m \geq 3, \text{ then } m|L_n \text{ iff } \frac{n}{y(m)} \text{ is an odd integer} \quad (6)$$

$$(L_n, L_{5n}/L_n) = 1 \quad (7)$$

$$\text{If } k \text{ is odd, then } (L_n, L_{kn}/L_n) | k. \quad (8)$$

$$\text{If } p \text{ is an odd prime, then } p|L_n \text{ iff } \frac{2n}{z(p)} \text{ is an odd integer.} \quad (9)$$

$$L_{2n} = L_n^2 - 2(-1)^n \quad (10)$$

$$L_n | L_{kn} \text{ iff } k \text{ is odd or } n=1. \quad (11)$$

$$L_{mn}/L_n = (-1)^{\frac{1}{2}(m-1)(n+1)} + \sum_{j=1}^{\frac{1}{2}(m-1)} (-1)^{(j-1)(n+1)} L_{(m+1-2j)n} \text{ if } m \text{ is odd.} \quad (12)$$

$$\text{If } p \text{ is odd and } p|L_n, \text{ then } 2 \leq y(p) \leq \frac{1}{2}(p+1). \quad (13)$$

$$\text{If } p \text{ and } m \text{ are odd, } p \nmid m, \text{ and } p^h | L_n, \text{ then } p^{h+k} | L_{mnp^k} \text{ for all } k \geq 0. \quad (14)$$

$$L_{2^n} \equiv 3 \pmod{4} \text{ for all } n \geq 1. \quad (15)$$

$$L_{8n} \equiv 2 \pmod{3} \text{ for all } n \geq 1. \quad (16)$$

$$\text{If } p \text{ is a prime such that } y(p) \text{ exists, then } (p, y(p)) = 1. \quad (17)$$

$$\text{If } m/(m,n) \text{ and } n/(m,n) \text{ are both odd, then } (L_m, L_n) = L_{(m,n)} \quad (18)$$

$$\text{If } y(p^2) = py(p), \text{ then } y(p^k) = p^{k-1}y(p) \text{ for all } k \geq 1. \quad (19)$$

Remarks: (1) follows from Theorem 11 in [2] with $a=1$; (4) is Theorem 1 in [1]; (8) follows from Theorem 4 in [5]; (12) follows from (44) in [4]; (14) follows from Theorem XI in [6]; (19) follows from (14). The other identities are elementary.

3. THE MAIN RESULTS

THEOREM 1 If p is a prime such that $y(p)$ exists and $L_{y(p)} = pu^2$, then (*) has the unique solution: $n = y(p)$, $x^2 = u^2$.

PROOF: This follows from hypothesis and (1).

THEOREM 2 If $p \in \{3, 7, 11, 19, 29, 47, 199, 521, 2207, 9349\}$, then (*) has a solution with $n = 2, 4, 5, 9, 7, 8, 11, 13, 16, 19$ respectively; if $p=19$, then $x^2=4$; in each other case, $x^2=1$.

PROOF: This follows from (2) and Theorem 1, since $L_2=3$, $L_4=7$, $L_5=11$, $L_9=19 \cdot 4$, $L_7=29$, $L_8=47$, $L_{11}=199$, $L_{13}=521$, $L_{16}=2207$, $L_{19}=9349$, and $y(19)=9$.

THEOREM 3 $L_{3k} = px^2$ iff either (i) $k=p=2$, $x^2=9$, or (ii) $k=3$, $p=19$, $x^2=4$.

PROOF: Sufficiency is readily shown, since $L_6=18=2(3)^2$ and $L_9=76=19(2)^2$. Now suppose $L_{3k} = px^2$. Let $d = (L_k, L_{3k}/L_k)$. If $k \not\equiv 2 \pmod{4}$, then (3) implies $d=1$, so (1) implies $L_k=u^2$, $L_{3k}/L_k = pv^2$ for some u, v . Now (4) implies $k=1$ or 3 . If $k=1$, then $L_3/L_1 = 4 = pv^2$, an impossibility. If $k=3$, then $L_9/L_3 = 19 = pv^2$,

so $p=19$ and $x^2 = L_9/19 = 4$. If $k \equiv 2 \pmod{4}$, then (3) implies $d=3$, so either (i) $L_k = 3u^2$, $L_{3k}/L_k = 3pv^2$, or (ii) $L_k = 3pu^2$, $L_{3k}/L_k = 3v^2$ for some u, v . If (i) holds, then Theorem 2 implies $k=2$, so $L_{3k} = L_6 = 18 = px^2$, which implies $p=2$ and $x^2=9$. If (ii) holds, then (5) implies $L_{3k}^2 - 3 = 3v^2$. Since $3|L_k$, we get $3v^2 \equiv -3 \pmod{9}$, so $v^2 \equiv -1 \pmod{3}$, an impossibility.

THEOREM 4 If $p > 19$ and $3|y(p)$, then $L_n = px^2$ is impossible.

PROOF: If $L_n = px^2$, then $p|L_n$, so (6) implies $y(p)|n$. Now hypothesis implies $3|n$, so $n=3k$ for some k . The conclusion now follows from hypothesis and Theorem 3.

THEOREM 5 (*) has no solution if $p \in \{23, 31, 79, 83, 107, 167, 181, 211, 227, 229, 241, 271, 349, 379, 383, 409, 431, 439, 443, 467, 499, 503, 541, 571, 587, 601, 631, 647, 683, 691, 739, 751, 769, 811, 827, 859, 863, 887, 919, 947, 983, 991\}$.

PROOF: This follows from hypothesis and Theorem 4, since in each case, $p > 19$ and according to [3], $3|y(p)$.

THEOREM 6 $L_{5k} = px^2$ iff $k = x^2 = 1$ and $p=11$.

PROOF: Sufficiency is readily shown, since $L_5 = 11 = 11 \cdot 1^2$. Now suppose $L_{5k} = px^2$. Theorem 2 of [1] implies p is odd. Now (7), (1) and hypothesis imply $L_k = u^2$, $L_{5k}/L_k = pv^2$ for some u, v . Now (4) implies $k=1$ or 3. If $k=1$, then $pv^2 = L_5/L_1 = 11$, so $p=11$ and $x^2 = L_5/11 = 1$. If $k=3$, then $pv^2 = L_{15}/L_3 = 1364/4 = 341 = 11 \cdot 31$, an impossibility.

THEOREM 7 If $L_n = px^2$ and $5|y(p)$, then $n=5$, $p=11$, $x^2=1$.

PROOF: Hypothesis and (6) imply $y(p)|n$. Therefore hypothesis implies $5|n$, that is, $n=5k$ for some k , so the conclusion follows from Theorem 6.

THEOREM 8 (*) has no solution if $p \in \{41, 71, 101, 131, 151, 191, 251, 311, 331, 401, 491, 641, 911, 941, 971\}$.

PROOF: This follows from Theorem 7, since in each case, $p > 11$, and according to [3], $5|y(p)$.

THEOREM 9 Let p be an odd prime such that $y(p)$ exists and is odd, and such that for every prime divisor, q , of $y(p)$, $z(q) \not\equiv 2 \pmod{4}$. If $L_n = px^2$, then $n = y(p)$.

PROOF: If $L_n = px^2$, then (6) implies $n = my(p)$ for odd m . Now (8) implies $d|\frac{n}{m}$, that is, $d|y(p)$. If $d > 1$, then there exists an odd prime, q , such that $q|d$. Therefore $q|L_m$, so (9) implies $\frac{2m}{z(q)}$ is an odd integer; since m is odd, this implies $z(q) \equiv 2 \pmod{4}$, contrary to hypothesis. Therefore $d=1$. Now (13) implies $y(p) > 1$, so $m < n$. Therefore hypothesis and (1) imply $L_m = u^2$, $L_n/L_m = pv^2$ for some u, v . Now (4) implies $m=1$ or 3. If $m=3$, then $L_n = p(2v)^2$, so Theorem 3 implies $n=9$, $p=19$. But then $\frac{n}{y(p)} = 1 \neq 3$. Therefore $m=1$ so $n = y(p)$.

THEOREM 10 (*) has no solution if $p \in \{139, 179, 239, 461, 509, 599, 619, 659\}$.

PROOF: This follows from hypothesis and Theorem 9, since in each case, according to [3] and [7], p fulfills the hypothesis of Theorem 9, yet $L_{y(p)} \neq px^2$.

In the work which follows, we will need the following lemmas:

$$\text{LEMMA 1} \quad L_{2^j t} \equiv \begin{cases} 2(-1)^{t+1} \pmod{L_t^2} & \text{if } j=1 \\ 2 & \pmod{L_t^2} & \text{if } j \geq 2 \end{cases}$$

PROOF: (Induction on j) (10) implies Lemma 1 holds for $j=1$. If $j \geq 2$, then (10) implies $L_{2^j t} = L_{2^{j-1} t}^2 - 2(-1)^{2^{j-1} t} = L_{2^{j-1} t}^2 - 2$. But $L_{2^{j-1} t}^2 \equiv 4 \pmod{L_t^2}$ by induction hypothesis. Therefore $L_{2^j t} \equiv 4 - 2 \equiv 2 \pmod{L_t^2}$.

$$\text{LEMMA 2} \quad \text{If } k \geq 1 \text{ and } 2|n, \text{ then } L_{2kn} \equiv 2(-1)^k \pmod{L_n^2}.$$

PROOF: Hypothesis and (10) imply $L_{2kn} = L_{kn}^2 - 2$. If k is odd, then (11) implies $L_n^2 | L_{kn}^2$, so $L_{2kn} \equiv -2 \equiv 2(-1)^k \pmod{L_n^2}$. If $k=2^j r$ with $j \geq 1$ and r odd, then Lemma 1 implies $L_{2kn} = L_{2^{j+1} r n} \equiv 2 \pmod{L_{rn}^2}$. Now (11) implies $L_n^2 | L_{rn}^2$, so $L_{2kn} \equiv 2 \equiv 2(-1)^k \pmod{L_n^2}$.

$$\text{LEMMA 3} \quad \text{If } m-1 \equiv n \equiv 0 \pmod{2}, \text{ then } L_{mn}/L_n \equiv m(-1)^{\frac{1}{2}(m-1)} \pmod{L_n^2}.$$

PROOF: Hypothesis and (12) imply $L_{mn}/L_n = (-1)^{\frac{1}{2}(m-1)} + \sum_{j=1}^{\frac{1}{2}(m-1)} (-1)^{j-1} L_{(m+1-2j)n}$. Hypothesis and Lemma 2 imply $L_{(m+1-2j)n} \equiv 2(-1)^{\frac{1}{2}(m+1-2j)} \pmod{L_n^2}$. Therefore $L_{mn}/L_n \equiv (-1)^{\frac{1}{2}(m-1)} + \sum_{j=1}^{\frac{1}{2}(m-1)} 2(-1)^{\frac{1}{2}(m-1)} \equiv (1+2(\frac{m-1}{2}))(-1)^{\frac{1}{2}(m-1)} \equiv m(-1)^{\frac{1}{2}(m-1)} \pmod{L_n^2}$.

$$\text{LEMMA 4} \quad \text{If } p \text{ is an odd prime, } p|L_n, \text{ and } 2|n, \text{ then } L_{pn}/pL_n \equiv (-1)^{\frac{1}{2}(p-1)} \pmod{p}.$$

PROOF: Hypothesis and Lemma 3 imply $L_{pn}/L_n \equiv p(-1)^{\frac{1}{2}(p-1)} \pmod{L_n^2}$. Now hypothesis implies $L_{pn}/L_n \equiv p(-1)^{\frac{1}{2}(p-1)} \pmod{p^2}$, from which the conclusion immediately follows.

$$\text{LEMMA 5} \quad \text{If } 2|n, p|L_n, p \text{ is prime, and } p \equiv 3 \pmod{4}, \text{ then } L_{pn}/pL_n \neq s^2.$$

PROOF: Hypothesis and Lemma 4 imply $L_{pn}/pL_n \equiv -1 \pmod{p}$. Also, hypothesis implies $s^2 \not\equiv -1 \pmod{p}$, so $L_{pn}/pL_n \neq s^2$.

$$\text{LEMMA 6} \quad \text{Let } L_n = px^2 \text{ where } p \text{ and } y(p) \text{ are odd. Let } m | \frac{n}{y(p)}.$$

Let $d = (L_m, L_n/L_m)$. Then $d=1$ iff $m=1$.

PROOF: If $m=1$, then $d|L_1$, that is $d|1$, so $d=1$. Conversely, if $d=1$, then since hypothesis and (13) imply $m < n$, hypothesis and (1) imply $L_m = u^2$, $L_n/L_m = pv^2$ for some u, v . Now (4) implies $m=1$ or 3 . If $m=3$, then hypothesis and Theorem 3 imply $p=19$, $n=9=y(19)$, so $m|1$, an impossibility. Therefore $m=1$.

LEMMA 7 If $L_n = px^2$, p and $y(p)$ are odd, $m > 1$ and $m \mid \frac{n}{y(p)}$, then $(L_m, L_n/L_m) > 1$.

PROOF: This follows from hypothesis and Lemma 6.

LEMMA 8 Let p, q be odd primes such that $pq \mid L_n$ for some n . Then $2^h \mid \mid y(p)$ iff $2^h \mid \mid y(q)$, where $h \geq 0$.

PROOF: Hypothesis and (6) imply $n = jy(p) = ky(q)$ with j, k odd. The conclusion now follows.

THEOREM 11 Let $L_n = px^2$, where p is an odd prime, $2^h \mid \mid y(p)$ for some $h \geq 1$, and $L_{2^h} = q$ is prime. Then either (i) $n=2^h$, $p=q$, $x^2=1$, or (ii) $n=2^h q$, $p = L_{2^h q} / (qt)^2$, $x^2 = (qt)^2$ for some $t \geq 1$.

PROOF: Hypothesis and (15) imply $q \equiv 3 \pmod{4}$, so q is odd. Hypothesis implies $y(p)/2^h$ is odd, so (11) implies $L_{2^h} \mid L_{y(p)}$, that is, $q \mid L_{y(p)}$. Hypothesis and (6) imply $n/y(p)$ is an odd integer, so (11) implies $L_{y(p)} \mid L_n$, hence $q \mid L_n$. If $p=q$, then hypothesis and (1) imply $n=2^h$, $x^2=1$. If $p \neq q$, then hypothesis implies $q \mid x^2$, so $q^2 \mid x^2$, so $q^2 \mid L_n$. Now (6) implies $n = my(q^2)$ for odd m . Hypothesis and (2) imply $y(q^2) = qy(q)$, so $n = mqy(q)$. We have $L_{mqy(q)} (L_{mqy(q)} / L_{my(q)}) = px^2$. Let $d = (L_{mqy(q)} / L_{my(q)})$. Now (8) implies $d \mid q$; (6) implies $q \mid L_{my(q)}$. Let $q^j \mid \mid L_{my(q)}$. Then (14) implies $q^{j+1} \mid L_{mqy(q)}$ so $q \mid (L_{mqy(q)} / L_{my(q)})$. Therefore $q \mid d$, so $d=q$. Letting $t = x/q$, we obtain $(L_{my(q)} / q) (L_{mqy(q)} / qL_{my(q)}) = pt^2$, where the factors on the left side of the equation are relatively prime. Therefore either (a) $L_{my(q)} / q = pu^2$, $L_{mqy(q)} / qL_{my(q)} = v^2$, or (b) $L_{my(q)} / q = u^2$, $L_{mqy(q)} / qL_{my(q)} = pv^2$ for some u, v . Now hypothesis and (2) imply $y(q) = 2^h$, so Lemma 5 implies (a) is impossible. Therefore (b) must hold. Now (1), (2) and hypothesis imply $m = u^2 = 1$, $L_{qy(q)} = p(qv)^2 = p(qt)^2$, $n = qy(q) = 2^h q$, $x=qt$.

THEOREM 12 If p is an odd prime and $2^h \mid \mid y(p)$ where $1 \leq h \leq 4$, then the only solutions of (*) are given by Table 1 below.

Table 1

n	p	x^2
2	3	1
4	7	1
8	47	1
16	2207	1
28	14503	49

PROOF: This follows from hypothesis and Theorem 11, since $L_2=3$, $L_4=7$, $L_8=47$, $L_{16}=2207$ (all primes); also $L_{28} = 14503 \cdot 7^2$. Note that $L_6/3^2 = 2$ (prime but not odd.) According to [7], $L_{376}/47^2 \neq pt^2$. According to the referee, $1553729 \nmid L_{35312}$ and $L_{35312}/1553729 \cdot 2207^2 \neq t^2$.

THEOREM 13 (*) has no solution if $p \in \{43, 67, 103, 163, 223, 263, 281, 283, 307, 347, 367, 449, 463, 487, 523, 547, 563, 569, 607, 643, 727, 743, 787, 823, 881, 883, 907, 929, 967\}$.

PROOF: This follows from hypothesis and Theorem 12, since in each case, according to [3], p satisfies the hypothesis of Theorem 12 but does not appear in Table 1 above.

THEOREM 14 $L_{11k} = px^2$ iff $k = x^2 = 1$ and $p=199$.

PROOF: Sufficiency is readily shown, since $L_{11} = 199 = 199 \cdot 1^2$. Now suppose $L_{11k} = px^2$. Let $d = (L_k, L_{11k}/L_k)$. (8) implies $d|11$. If $d=11$, then since $y(11)=5$, (6) implies $5|k$, so $5|11k$. But then hypothesis and Theorem 6 imply $11k=5$, an impossibility. If $d=1$, then (10) implies $L_k = u^2$, $L_{11k}/L_k = pv^2$ for some u, v . Now (4) implies $k=1$ or 3. Theorem 3 implies $k \neq 3$, so $k=1$, hence $p=199$, $x^2=1$.

THEOREM 15 If $L_n = px^2$ and $11|y(p)$, then $n=11$, $p=199$, $x^2=1$.

PROOF: Hypothesis and (6) imply $11|n$, so the conclusion follows from Theorem 14.

THEOREM 16 $L_n = 419x^2$ is impossible.

PROOF: According to [3], $y(419) = 209 = 11 \cdot 19$. The conclusion now follows from Theorem 15.

THEOREM 17 $L_n = 127x^2$ is impossible.

PROOF: Suppose $L_n = 127x^2$. Since $y(127)=64$, (6) implies $64|n$. Hypothesis and (16) now imply $x^2 \equiv 2 \pmod{3}$, an impossibility.

THEOREM 18 If p and $y(p)=q$ are primes, $q > 3$, $q^2 \nmid L_{y(q)}$, $p^2 \nmid L_q$, $L_q \neq ps^2$, and either (I) $2|y(q)$ or (II) $2 \nmid y(q)$ and the equation $L_m = qs^2$ (considered as an equation in m) either (A) has no solution or (B) has the solution $m=y(q)$ but there exists a prime, t , such that $t \nmid (L_q/p)$ and $t \nmid y(q)$, then $L_n = px^2$ is impossible.

PROOF: Suppose $L_n = px^2$. Hypothesis and (6) imply $n=mq$, m odd, $m > 1$. Let $d = (L_m, L_n/L_m)$. (8) implies $d|q$. Lemma 7 implies $d > 1$, so $d=q$. Therefore $q|L_n$. If (I) holds, then we get a contradiction via Lemma 8, since $pq|L_n$. If (II) holds, then either (i) $L_m = qu^2$, $L_n/L_m = pqv^2$ or (ii) $L_m = pqu^2$, $L_n/L_m = qv^2$ for some u, v . If (i) holds, then (1) and hypothesis imply $m=y(q)$. Now (B) implies there exists a prime, t , such that $t \nmid (L_q/p)$ and $t \nmid y(q)$. If $t=p$, then $p \nmid (L_q/p)$, so $p^2 \nmid L_q$, contrary to hypothesis. If $t \neq p$, then $t \nmid L_q$, so (14) implies $t \nmid L_{qy(q)}$, that is, $t \nmid px^2$, so $t \nmid x^2$, an impossibility.

If (ii) holds instead, then (6) implies $y(p)|m$ and $y(q)|m$, so $\text{LCM}(y(p), y(q))|m$, that is, $\text{LCM}(q, y(q))|m$. But (17) implies $\text{LCM}(q, y(q))=qy(q)$, so $qy(q)|m$. Since $q^2 \nmid L_{y(q)}$ by hypothesis, we have $q \nmid L_{y(q)}$, so (14) implies $y(q^2)=qy(q)$, hence $y(q^2)|m$. Therefore hypothesis and (6) imply $q^2|L_m$, so that $q|u^2$, which implies $q^2|u^2$, hence $q^3|L_m$. Now (19) and (6) imply $q^2 y(q)|m$, so $m=qk$, $qy(q)|k$. Let $d_1 = (L_k, L_m/L_k)$. Now (8) implies $d_1|q$. Therefore $L_k = ca^2$, where $c = 1, p, q$, or pq . Since $k < m < n$, (1) implies $c \neq p$, $c \neq pq$. If $c=1$, then (4) implies $k=1$ or 3, violating $qy(q)|k$. If $c=q$, then hypothesis and (1) imply $k=y(q)$, again violating $qy(q)|k$.

THEOREM 19 (*) has no solution if $p \in \{59, 359, 479, 709, 719, 809, 839\}$.

PROOF: In each case, according to [3] and [7], p satisfies the hypothesis of Theorem 18, from which the conclusion follows. Table 2 below gives the details.

Table 2

p	q	y(q)	relevant section of Theorem 18
59	29	7	II B, $t=19489$
359	179	89	II A (see Theorem 10 above)
479	239	119	II A (see theorem 10 above)
709	59	29	II A (see first entry in Table 2)
719	359	179	II A (see second entry in Table 2)
809	101	50	I
839	419	209	II A (see Theorem 16 above)

We summarize our results in Table 3 below, which contains all solutions of (*) with $2 < p < 1000$.

Table 3

p	n	x^2
3	2	1
7	4	1
11	5	1
19	9	4
29	7	1
47	8	1
199	11	1
521	13	1

Remark: The related results of M. Goldman [8] follow immediately from (1).

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