ON STABILITY OF ADDITIVE MAPPINGS

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ABSTRACT. In this paper we answer a question of Th. M. Rassias concerning an extension of validity of his result proved in [3].

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1. INTRODUCTION.

In connection with a problem posed by Ulam (cf. [5]; see also [2]) Th. M. Rassias [3] proved the following theorem on stability of linear mappings in Banach spaces.

THEOREM 1. (see [3]) Let E_1 and E_2 be two (real) Banach spaces and let $f: E_1 \rightarrow E_2$ be a mapping such that for each fixed $x \in E_1$ the transformation $\mathbb{R} \ni t \rightarrow f(tx)$ is continuous. Moreover, assume that there exist $\varepsilon \in [0, \infty)$ and $p \in [0, 1]$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (1.1)$$

for all $x, y \in E_1$. Then there exists a unique linear mapping $T: E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \delta \|x\|^p \quad (1.2)$$

for all $x \in E_1$, where $\delta = \frac{2\varepsilon}{2-2^p}$.

As was mentioned by Th. M. Rassias [4], the proof presented in [3] reveals that, in fact, it works for every p from the interval $(-\infty, 1)$ and, therefore, the theorem holds true for all such p 's. It is also readily seen that the only purpose of assuming that all the transformations of the form $t \rightarrow f(tx)$ are continuous is to guarantee the real homogeneity of the mapping T . Without this assumption one can show that f is approximated by an additive mapping T which means that T satisfies the following equation

$$T(x+y) = T(x) + T(y) \quad (1.3)$$

for all $x, y \in E_1$. Finally, it should be noticed that the completeness of the space E_1 may be removed from the assumptions of Theorem 1. However, there is still one non-trivial (as it seems) question concerning a possible extension of the range of validity of Theorem 1. Namely, one can ask whether the same result holds true under the hypothesis that p is taken from the interval $[1, \infty)$ (obviously in this case the constant δ should have been defined in a different manner). Such a

problem was raised by Th. M. Rassias during the 27th International Symposium on Functional Equations which was held in Bielsko-Biala, Katowice and Krokow in August 1989. The goal of the present note is to give a complete solution to this problem.

2. MAIN RESULTS.

First, let us realize why the proof of Theorem 1 in its original form (see [3]) does not work for $p \geq 1$. The fundamental role in this proof is played by the sequence

$$\left\{ \frac{1}{2^n} f(2^n x) : n \in \mathbb{N} \right\} \quad (2.1)$$

which, under the assumptions of Theorem 1 (in fact as long as $p \in (-\infty, 1)$) is convergent for each fixed $x \in E_1$. Then $T: E_1 \rightarrow E_2$ defined by the formula

$$T(x) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n x), \quad x \in E_1 \quad (2.2)$$

is the desired linear mapping approximating f . The argument ensuring the convergence of sequence (2.1) is no longer valid when p becomes greater or equal to 1, so in order to carry the proof over to this case, one has to change the argument itself or the definition of the mapping T . It turns out that, for $p > 1$, the latter modification of the proof is possible. As a result we obtain the following extension of Theorem 1:

THEOREM 2. Let E_1 and E_2 be two (real) normed linear spaces and assume that E_2 is complete. Let $f: E_1 \rightarrow E_2$ be a mapping for which there exist two constants $\varepsilon \in [0, \infty)$ and $p \in \mathbb{R} \setminus \{1\}$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (2.3)$$

for all $x, y \in E_1$. Then there exists a unique additive mapping $T: E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \delta \|x\|^p \quad (2.4)$$

for all $x \in E_1$, where

$$\delta = \begin{cases} \frac{2\varepsilon}{2-2^p} & \text{for } p < 1, \\ \frac{2\varepsilon}{2^p-2} & \text{for } p > 1. \end{cases}$$

Moreover, if for each $x \in E_1$ the transformation $\mathbb{R} \ni t \rightarrow f(tx)$ is continuous, then the mapping T is linear.

PROOF. In view of what has been said so far, it remains to consider the case $p > 1$. The main innovation in comparison with the case $p < 1$ consists in defining the mapping T by the formula

$$T(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right), \quad x \in E_1 \quad (2.5)$$

instead of (2.2). Obviously, one has to verify the convergence of the sequence occurring on the right-hand side of (2.5).

Putting $\frac{x}{2}$ in place of x and y in inequality (2.3), we obtain

$$\|f(x) - 2f\left(\frac{x}{2}\right)\| \leq 2\varepsilon \left\|\frac{x}{2}\right\|^p = 2^{1-p}\varepsilon \|x\|^p$$

for all $x \in E_1$. Hence for each $n \in \mathbb{N}$ and every $x \in E_1$, we have

$$\begin{aligned} \|f(x) - 2^n f\left(\frac{x}{2^n}\right)\| &\leq \|f(x) - 2f\left(\frac{x}{2}\right)\| + 2\|f\left(\frac{x}{2}\right) - 2f\left(\frac{x}{2^2}\right)\| + \dots + 2^{n-1}\|f\left(\frac{x}{2^{n-1}}\right) - 2f\left(\frac{x}{2^n}\right)\| \\ &\leq 2^{1-p}\varepsilon \|x\|^p + 2 \cdot 2^{1-p}\varepsilon \left\|\frac{x}{2}\right\|^p + \dots + 2^{n-1} \cdot 2^{1-p}\varepsilon \left\|\frac{x}{2^{n-1}}\right\|^p \\ &= (2^{1-p} + 2^{2(1-p)} + \dots + 2^{n(1-p)})\varepsilon \|x\|^p \end{aligned}$$

$$\leq \delta \|x\|^p, \quad (2.6)$$

where δ is the sum of the following convergent series:

$$\sum_{n=1}^{\infty} 2^{n(1-p)} \varepsilon = \frac{2\varepsilon}{2^p - 2}.$$

Now, fix an $x \in E_1$ and choose arbitrary $m, n \in \mathbb{N}$ such that $m > n$. Then

$$\begin{aligned} \|2^m f(\frac{x}{2^m}) - 2^n f(\frac{x}{2^n})\| &= 2^n \|2^{m-n} f(\frac{1}{2^{m-n}} \cdot \frac{x}{2^n}) - f(\frac{x}{2^n})\| \\ &\leq 2^n \delta \|\frac{x}{2^n}\|^p = 2^{n(1-p)} \delta \|x\|^p, \end{aligned}$$

which becomes arbitrarily small as $n \rightarrow \infty$. On account of the completeness of the space E_2 , this implies that the sequence $\{2^n f(\frac{x}{2^n}) : n \in \mathbb{N}\}$ is convergent for each $x \in E_1$. Thus T is correctly defined by (2.5). Moreover, it satisfies condition (2.4) which results on letting $n \rightarrow \infty$ in (2.6).

Finally, replacing x by $\frac{x}{2^n}$ and y by $\frac{y}{2^n}$ in (2.3) and then multiplying both sides of the resulting inequality by 2^n , we get

$$\|2^n f(\frac{x+y}{2^n}) - 2^n f(\frac{x}{2^n}) - 2^n f(\frac{y}{2^n})\| \leq 2^{n(1-p)} \varepsilon (\|x\|^p + \|y\|^p),$$

for $x, y \in E_1$. Since the right-hand side of this inequality tends to zero as $n \rightarrow \infty$, it becomes apparent that the mapping T defined by (2.5) is additive.

The proof of the homogeneity of T (under the supplementary assumption that $t \rightarrow f(tx)$ is continuous for each $x \in E_1$) needs no essential alterations in comparison with the case $p < 1$. It is also clear what has to be changed in the proof of the uniqueness of T .

Theorem 2 leaves the case $p = 1$ undecided. This is not a mere coincidence. It turns out that 1 is the only critical value of p to which Theorem 2 can not be extended. In fact, we shall show that $\varepsilon > 0$ one can find a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$|f(x+y) - f(x) - f(y)| \leq \varepsilon(|x| + |y|) \quad (2.7)$$

for all $x, y \in \mathbb{R}$, but, at the same time, there is no constant $\delta \in [0, \infty)$ and no additive function $T: \mathbb{R} \rightarrow \mathbb{R}$ satisfying the condition

$$|f(x) - T(x)| \leq \delta |x| \quad \text{for all } x \in \mathbb{R}, \quad (2.8)$$

This singularity is illustrated by the following:

EXAMPLE. Fix $\varepsilon > 0$ and put $\mu = \frac{\varepsilon}{6}$. First we define a function $\phi: \mathbb{R} \rightarrow \mathbb{R}$ by

$$\phi(x) = \begin{cases} \mu & \text{for } x \in [1, \infty), \\ \mu x & \text{for } x \in (-1, 1), \\ -\mu & \text{for } x \in (-\infty, -1]. \end{cases}$$

Evidently, ϕ is continuous and $|\phi(x)| \leq \mu$ for all $x \in \mathbb{R}$. Therefore, a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is correctly defined by the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n}, \quad x \in \mathbb{R}.$$

Since f is defined by means of a uniformly convergent series of continuous functions, f itself is continuous. Moreover,

$$|f(x)| \leq \sum_{n=0}^{\infty} \frac{\mu}{2^n} = 2\mu, \quad x \in \mathbb{R}.$$

We are going to show that f satisfies (2.7).

If $x = y = 0$, then (2.7) is trivially fulfilled. Next assume that $0 < |x| + |y| < 1$. Then there exists an $N \in \mathbb{N}$ such that

$$2^N \leq |x| + |y| < \frac{1}{2^{N-1}}.$$

Hence, $|2^N - 1|x| < 1$, $|2^N - 1|y| < 1$ and $|2^N - 1|(x + y)| \leq 2^N - 1(|x| + |y|) < 1$, which implies that for each $n \in \{0, 1, \dots, N-1\}$ the numbers $2^n x$, $2^n y$ and $2^n(x + y)$ remain in the interval $(-1, 1)$. Since ϕ is linear on this interval, we infer that

$$\phi(2^n(x + y)) - \phi(2^n x) - \phi(2^n y) = 0$$

for $n = 0, 1, \dots, N-1$. As a result, we get

$$\begin{aligned} \frac{|f(x + y) - f(x) - f(y)|}{(|x| + |y|)} &\leq \sum_{n=N}^{\infty} \frac{|\phi(2^n(x + y)) - \phi(2^n x) - \phi(2^n y)|}{2^n(|x| + |y|)} \\ &\leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k 2^N(|x| + |y|)} \leq \sum_{k=0}^{\infty} \frac{3\mu}{2^k} = 6\mu = \varepsilon. \end{aligned}$$

Finally, assume that $|x| + |y| \geq 1$. Then merely by virtue of the boundedness of f we have

$$\frac{|f(x + y) - f(x) - f(y)|}{|x| + |y|} \leq 6\mu = \varepsilon.$$

Thus we conclude that f satisfies (2.7) for all real x and y .

Now, contrary to what we claim, suppose that there exist a $\delta \in [0, \infty)$ and an additive function $T: \mathbb{R} \rightarrow \mathbb{R}$ such that (2.8) holds true. Hence, from the continuity of f it follows that T is bounded on some neighbourhood of zero. Then, by a classical result (see e.g. [1], 2.1.1., Theorem 1) there exists a real constant c such that

$$T(x) = cx, \quad x \in \mathbb{R}$$

Hence,

$$|f(x) - cx| \leq \delta |x|, \quad x \in \mathbb{R},$$

which implies that

$$\left| \frac{f(x)}{x} \right| \leq \delta + |c|, \quad x \in \mathbb{R}.$$

On the other hand, we can choose an $N \in \mathbb{N}$ so large that $N\mu > \delta + |x|$. Then picking out an x from the interval $(0, \frac{1}{2^{N-1}})$, we have $2^n x \in (0, 1)$ for each $n \in \{0, 1, \dots, N-1\}$. Consequently, for such an x we have

$$\frac{f(x)}{x} \geq \sum_{n=0}^{\infty} \frac{\phi(2^n x)}{2^n x} = \sum_{n=0}^{\infty} \frac{\mu 2^n x}{2^n x} = N\mu > \delta + |x|,$$

which yields a contradiction. Thus the function f provides a good example to the effect that Theorem 2 fails to hold for $p = 1$.

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