ON POLYNOMIAL EXPANSION OF MULTIVALENT FUNCTIONS

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ABSTRACT. Coefficient bounds for mean p-valent functions, whose expansion in an ellipse has a Jacobi polynomial series, are given in this paper.

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1. INTRODUCTION.

Let \( E_0 = \{ z = \cosh(s + it), \ 0 < t < 2\pi, \ s_o = \tanh^{-1}(b/a), \ a > b > 0 \} \) be a fixed ellipse whose foci are \( \pm 1 \). Let also \( r = a + b \) be the sum of the semi-axes of \( E_0 \). It is known (Szegő [1], Theorem 9.1.11, see also p. 245) that a function \( f(z) \) which is regular in \( \text{Int}(E_0) \) (this means the interior of \( E_0 \)) has an expansion of the form

\[
f(z) = \sum_{n=0}^{\infty} a_n P_n^{(a,b)}(z)
\]

where here and throughout this paper \( a, b > -1 \). This expansion converges locally uniformly in \( \text{Int}(E_0) \). In [2] the author has given some coefficient bounds for functions mean p-valent and has an expansion in terms of Chebyshev polynomials in \( \text{Int}(E_0) \). Such polynomials are generated by the special case \( a = 1 \) in Jacobi polynomials. Other special cases of interest are the Legendre and the altraspherical polynomials generated by \( a = \beta = 0 \) and \( a = \beta \) respectively [1, p. 80-89].

In this paper we generalize results given in [2] to functions of the form (1.1) and mean p-valent in \( \text{Int}(E_0) \). In view of [2] we call \( f(z) \) mean p-valent in \( \text{Int}(E_0) \) if

\[
W(R,f) = \left( \frac{1}{2\pi} \right) \int_0^{2\pi} \left[ n(\rho e^{i\phi}, f, \text{Int}(E_0)) \right] d\rho d\phi < pR^2
\]

where \( 0 < R < \infty \) and \( n(\rho e^{i\phi}, f, \text{Int}(E_0)) \) denotes the number of roots of the equation \( f(z) = w \) in Interior \( E_0 \), multiplicity being take into account.

We first recall from [2]:

THEOREM A. Let \( f(z) \) be mean p-valent in \( \text{Int}(E_0) \). Then for \( z = \cosh(s + it), \ \exp(s) = r \) and \( 1 < r < r_o \) we have

\[
|f(z)| = O(1) \ (1-r/r_o)^{-2p}
\]

where \( O(1) \) depends on \( a, b \) and \( f \) only.
THEOREM B. Let \( f(z) \) be mean \( p \)-valent in \( \text{Int}(E_0) \) and \( M(r,f) < C(1-r/r_0)^{-\gamma} \) where \( c, \gamma > 0 \) and \( M(r,f) = \max\{|f(z)|: z \in \text{Int}(E_0)\} \). Set \( z = \cosh(s+it) \), \( \exp(s) = r \), \( 1 < r < r_0 \) and

\[
I_1(r,f') = \frac{2\pi}{0} \int |f'(|\cosh(s+it)||\sinh(s+it)|dt.
\]

Then as \( r \to r_0 \) we have

\[
I_1(r,f') \to \begin{cases} 
0(1) \frac{(1-r/r_0)^{-\gamma}}{\gamma} & (\gamma > 1/2), \\
0(1) \frac{(1-r/r_0)^{-1/2}}{\log(1/(1-r/r_0))} & (\gamma = 1/2), \\
o(1) \frac{(1-r/r_0)^{-1/2}}{\log(1/(1-r/r_0))} & (\gamma < 1/2),
\end{cases}
\]

where \( O(1) \) and \( o(1) \) depend on \( a, b, \gamma \) and \( f \) only.

PROOF OF THEOREM B. Using Schwarz's inequality we have

\[
I_1(r,f') \leq \frac{2\pi}{0} \int |f'(|\cosh(s+it)||f(\cosh(s+it))|^{\lambda-2}|\sinh(s+it)|^{\lambda-1}dt, \quad 0 < \lambda < 2.
\]

Theorem B now follows in the same way as estimating inequality (14) of [2] by using [2, Lemmas 3 and 4].

We now need a suitable coefficient formula.

LEMMA 1.1. Let \( f(z) = \sum \frac{a_n}{n!} (z) \) be regular in \( \text{Int}(E_0) \) and

\[ E = \{z = \cosh(s+it), \quad 0 < \tau < 2\pi\}. \]

Then for a fixed \( s \) so that \( 0 < s < s_0 \) we have

\[
a_n = K_n(a,\beta)/h_n(a,\beta) \int_{E} \frac{f(z)}{n!} dz, \quad (n > 0), \quad (1.2)
\]

\[
\frac{1}{2(n+\alpha+1)} a_n = K_n^\alpha(\alpha+1,\beta+1)/h_n(\alpha+1,\beta+1) \int_{E} \frac{f(z)}{n!} dz, \quad (n > 1) \quad (1.3)
\]

where \( K_n(a,\beta) = 2^{n+\alpha+\beta+1} n(\alpha+1)\Gamma(n+\beta+1)/(2n+\alpha+\beta+2) \) and

\[
h_n(a,\beta) = 2^{\alpha+\beta+1} n(\alpha+1)\Gamma(n+\beta+1)/(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(\alpha+\beta+1).
\]

We note here, using Stirling's formula from Titmarsh [3, p. 57], that

\[
K_n(a,\beta)/h_n(a,\beta) = O(1)n^{1/2}/2^n \quad (1.4)
\]

as \( n \to \infty \), where \( O(1) \) depends on \( a, \beta \) only.

PROOF OF LEMMA. We have from [1, p. 245] that

\[
a_n = \text{Res}_{E} (z-1)^a (z+1)^\beta Q_n^\alpha(a,\beta)(z)f(z)dz. \quad (1.5)
\]

where \( n = 0,1,2,\ldots \).

We now see from [1, Theorem 4.6.2], (see also Erdelyi, Magnus, Oberhettinger and Tricomi [4, p. 171], and Freud [5, p. 44] that

\[
(z-1)^a (z+1)^\beta Q_n^\alpha(a,\beta)(z) = (1/2) \sum_{k=0}^{\infty} \frac{1}{k!} \int_{-1}^{1} (1-t)^a (1+t)^\beta k(t) dt.
\]

We note here, using Stirling's formula from Titmarsh [3, p. 57], that

\[
k_n(a,\beta)/2n+1 \quad (1.6)
\]
where $k_n^{(a, b)}$ is as defined above. In connection with this, see the argument used in the proof of formula (4.3.3) of [1, p.67].

Using (1.6) in (1.5) we immediately deduce (1.2).

Now differentiating (1.1) we see from equation (4.21.7) of [II that

$$f'(z) = \frac{1}{2(n+a+b+1)} a_n P_n^{(a+1, b+1)}(z).$$

Again, as in the proof of (1.2), we deduce from this and [1, p. 245] for $n > 1$, that

$$a_n = \frac{\pi}{(n-1)!} \left\{ (z-1)^{(n-1)/(n+1)} (z+1)^{(n-1)/(n-1)} \right\} (1/2^n) \int_{E} f'(z) dz$$

where we have used the equation $(z-1)^{(n-1)/(n+1)} (z+1)^{(n-1)/(n-1)} = k_n^{(a+1, b+1)} 2^n$. This is equation (1.3) and the proof of the lemma is now complete.

2. MAIN THEOREM.

THEOREM 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n P_n^{(a,b)}(z)$ be mean $p$-valent in $\text{Int}(E)$ and

$$M(r,f) \leq C(1-r/r_0)^{-\gamma}$$

where $C$, $\gamma > 0$ and $M(r,f)$ is as defined above. Then, as $n \to \infty$ we have

$$|a_n| = r_0^{-n}$$

where $O(1)$ and $o(1)$ depend on $a, b, a, b, \gamma$ and $f$ only.

PROOF OF THEOREM 2.1. From (1.3) and Theorem B we deduce, using the bounds

$$|\sinh(s+it)| \geq |\sinh s|, \quad |\cosh(s+it)| \leq |\cosh s|$$

that

$$\frac{1}{2(n+a+b+1)} a_n \leq \frac{(a+1, b+1)}{(n-1)!} \int_{E} (\cosh 1/(r^{1/2}))(\cosh I_1(r,f)/\sinh^n s)$$

where we have chosen $r = ((n-1)/n)r_0$ and provided that $1-n/(n-1)r_0 > 0$. This completes the proof of Theorem 2.1.

COROLLARY 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n P_n^{(a,b)}(z)$ be mean $p$-valent in $\text{Int}(E)$. Then, as $n \to \infty$ we have

$$|a_n| = r_0^{-n}$$

where $O(1)$ and $o(1)$ depend on $a, b, a, b, \gamma$ and $f$ only.
where $O(1)$ and $o(1)$ depend on $a, b, \alpha, \beta, p$ and $f$ only. In view of Theorem A, the proof of Corollary 2.1 follows by setting $\gamma = 2p$ in Theorem 2.1.

**COROLLARY 2.2.** Let $f(z) = \sum_{n=0}^{\infty} a_n (\alpha, \beta)(z)$ be univalent in $\text{Int}(E_o)$. Then as $n \to \infty$ we have

$$|a_n| = O(1)n^{3/2}\frac{r_o^{-n}}{r_o},$$

where $O(1)$ depends on $a, b, \alpha, \beta$ and $f$ only.

This corollary follows upon setting $p = 1$ in Corollary 2.1.

**REMARK.** Using the formula (4.21.2) of [1] and the argument used in [2, Remark 2] we see by setting $z = \xi \cosh s_o$ where $|\xi| = |\cos \tau + i \tanh \tau| < 1$ that

$$f(\xi \cosh s_o) = \sum_{n=0}^{\infty} \frac{\Gamma(2n+\alpha+\beta+1)}{n! \Gamma(n+\alpha+\beta+1)} a_n \left(\frac{\cosh s}{2^\alpha}\right)^n ((\xi - 1/\cosh s_o)^n$$

$$+ c_1 (\xi - 1/\cosh s_o)^{n-1} + \ldots + c_n / \cosh^n s_o)$$

and

$$\sum_{n=0}^{\infty} \hat{a}_n \tilde{p}(\alpha, \beta)(\xi)$$

where

$$\tilde{p}(\alpha, \beta)(\xi) = (\xi - 1/\cosh s_o)^n + c_1 (\xi - 1/\cosh s_o)^{n-1} + \ldots + c_n / \cosh^n s_o$$

and

$$\hat{a}_n = \frac{\Gamma(2n+\alpha+\beta+1)}{n!} \cosh^n s_o / 2^n \Gamma(n+1) \Gamma(n+\alpha+\beta+1).$$

Using this and Stirling's formula and letting $r_o \to \infty$ we see that Theorem 2.1 and Corollaries 2.1 and 2.2 correspond to analogous results for the unit disk (see Hayman [6]).

**REFERENCES**
