ON POLYNOMIAL EXPANSION OF MULTIVALENT FUNCTIONS

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ABSTRACT. Coefficient bounds for mean p -valent functions, whose expansion in an ellipse has a Jacobi polynomial series, are given in this paper.

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1. INTRODUCTION.

Let $E_0 = \{z = \cosh(s_0 + i\tau), 0 < \tau < 2\pi, s_0 = \tanh^{-1}(b/a), a > b > 0\}$ be a fixed ellipse whose foci are ± 1 . Let also $r_0 = a+b$ be the sum of the semi-axis of E_0 . It is known (Szegő [1], Theorem 9.1.1], see also p. 245) that a function $f(z)$ which is regular in $\text{Int}(E_0)$ (this means the interior of E_0) has an expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha, \beta)}(z) \quad (1.1)$$

where here and throughout this paper $\alpha, \beta > -1$. This expansion converges locally uniformly in $\text{Int}(E_0)$. In [2] the author has given some coefficient bounds for functions mean p -valent and has an expansion in terms of Chebyshev polynomials in $\text{Int}(E_0)$. Such polynomials are generated by the special case $\alpha = \beta = -1/2$ in Jacobi polynomials. Other special cases of interest are the Legendre and the ultraspherical polynomials generated by $\alpha = \beta = 0$ and $\alpha = \beta$ respectively [1, p. 80-89].

In this paper we generalize results given in [2] to functions of the form (1.1) and mean p -valent in $\text{Int}(E_0)$. In view of [2] we call $f(z)$ mean p -valent in $\text{Int}(E_0)$ if

$$W(R, f) = (1/\pi) \int_0^R \int_0^{2\pi} n(\rho e^{i\phi}, f, \text{Int}(E_0)) \rho d\rho d\phi < pR^2$$

where $0 < R < \infty$ and $n(\rho e^{i\phi}, f, \text{Int}(E_0))$ denotes the number of roots of the equation $f(z) = w$ in Interior E_0 , multiplicity being taken into account.

We first recall from [2]:

THEOREM A. Let $f(z)$ be mean p -valent in $\text{Int}(E_0)$. Then for $z = \cosh(s+i\tau)$, $\exp(s) = r$ and $1 < r < r_0$ we have

$$|f(z)| = O(1) (1-r/r_0)^{-2p}$$

where $O(1)$ depends on a, b and f only.

THEOREM B. Let $f(z)$ be mean p -valent in $\text{Int}(E_0)$ and $M(r, f) < C(1-r/r_0)^{-\gamma}$ where $c, \gamma > 0$ and $M(r, f) = \max\{|f(z)| : z \in \text{Int}(E_0)\}$. Set $z = \cosh(s+i\tau)$, $\exp(s)=r$, $1 < r < r_0$ and

$$I_1(r, f') = (1/2\pi) \int_0^{2\pi} |f'(\cosh(s+i\tau))| |\sinh(s+i\tau)| d\tau.$$

Then as $r \rightarrow r_0$ we have

$$I_1(r, f') = \begin{cases} O(1) (1-r/r_0)^{-\gamma}, & (\gamma > 1/2), \\ O(1) (1-r/r_0)^{-1/2} \log(1/(1-r/r_0)), & (\gamma = 1/2), \\ o(1) (1-r/r_0)^{-1/2}, & (\gamma < 1/2), \end{cases}$$

where $O(1)$ and $o(1)$ depend on a, b, γ and f only.

PROOF OF THEOREM B. Using Schwarz's inequality we have

$$\begin{aligned} I_1(r, f') &< [(1/2\pi) \int_0^{2\pi} |f'(\cosh(s+i\tau))|^2 |f(\cosh(s+i\tau))|^{\lambda-2} |\sinh(s+i\tau)|^2 d\tau]^{1/2} \\ &\times [(1/2\pi) \int_0^{2\pi} |f(\cosh(s+i\tau))|^{2-\lambda} d\tau]^{1/2} \end{aligned}$$

where $0 < \lambda < 2$. Theorem B now follows in the same way as estimating inequality (14) of [2] by using [2, Lemmas 3 and 4].

We now need a suitable coefficient formula.

LEMMA 1.1. Let $f(z) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha, \beta)}(z)$ be regular in $\text{Int}(E_0)$ and

$E = \{z = \cosh(s+i\tau), 0 \leq \tau < 2\pi\}$. Then for a fixed s so that $0 < s < s_0$ we have

$$a_n = (K_n^{(\alpha, \beta)} / h_n^{(\alpha, \beta)}) (1/2\pi i) \int_E \frac{f(z)}{z^{n+1}} dz, \quad (n \geq 0), \quad (1.2)$$

$$\frac{1}{2}(n+\alpha+\beta+1)a_n = (K_n^{(\alpha+1, \beta+1)} / h_{n-1}^{(\alpha+1, \beta+1)}) (1/2\pi i) \int_E \frac{f'(z)}{z^n} dz, \quad (n \geq 1) \quad (1.3)$$

where $K_n^{(\alpha, \beta)} = 2^{n+\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1) / \Gamma(2n+\alpha+\beta+2)$ and

$$h_n^{(\alpha, \beta)} = 2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1) / ((2n+\alpha+\beta+1) \Gamma(n+1) \Gamma(n+\alpha+\beta+1)).$$

We note here, using Stirling's formula from Titmarsh [3, p. 57], that

$$K_n^{(\alpha, \beta)} / h_n^{(\alpha, \beta)} = O(1) n^{1/2} / 2^n \quad (1.4)$$

as $n \rightarrow \infty$, where $O(1)$ depends on α, β only.

PROOF OF LEMMA. We have from [1, p. 245] that

$$a_n = \{\pi i h_n^{(\alpha, \beta)}\}^{-1} \int_E (z-1)^{\alpha} (z+1)^{\beta} Q_n^{(\alpha, \beta)}(z) f(z) dz \quad (1.5)$$

where $n = 0, 1, 2, \dots$.

We now see from [1, Theorem 4.61.2], (see also Erdelyi, Magnus, Oberhettinger and Tricomi [4, p. 171], and Freud [5, p. 44] that

$$\begin{aligned} (z-1)^{\alpha} (z+1)^{\beta} Q_n^{(\alpha, \beta)}(z) &= (1/2) \sum_{k=0}^{\infty} \frac{1}{z^{k+1}} \int_{-1}^1 (1-t)^{\alpha} (1+t)^{\beta} t^k p_n^{(\alpha, \beta)}(t) dt \\ &= K_n^{(\alpha, \beta)} / 2z^{n+1}, \end{aligned} \quad (1.6)$$

where $K_n^{(\alpha, \beta)}$ is as defined above. In connection with this, see the argument used in the proof of formula (4.3.3) of [1, p.67].

Using (1.6) in (1.5) we immediately deduce (1.2).

Now differentiating (1.1) we see from equation (4.21.7) of [1] that

$$f'(z) = \sum_{n=1}^{\infty} \frac{1}{2} (n+\alpha+\beta+1) a_n p_n^{(\alpha+1, \beta+1)}(z).$$

Again, as in the proof of (1.2), we deduce from this and [1, p. 245] for $n > 1$, that

$$\begin{aligned} \frac{1}{2} (n+\alpha+\beta+1) a_n &= \{ \pi i h_{n-1}^{(\alpha+1, \beta+1)} \}^{-1} \int_E (z-1)^{\alpha+1} (z+1)^{\beta+1} Q_{n-1}^{(\alpha+1, \beta+1)}(z) f'(z) dz \\ &= (K_{n-1}^{(\alpha+1, \beta+1)} / h_{n-1}^{(\alpha+1, \beta+1)}) (1/2\pi i) \int_E \frac{f'(z)}{z^n} dz \end{aligned}$$

where we have used the equation $(z-1)^{\alpha+1} (z+1)^{\beta+1} Q_{n-1}^{(\alpha+1, \beta+1)}(z) = K_{n-1}^{(\alpha+1, \beta+1)} / 2z^n$

which is deduced as in (1.6). This is equation (1.3) and the proof of the lemma is now complete.

2. MAIN THEOREM.

THEOREM 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha, \beta)}(z)$ be mean p -valent in $\text{Int}(E_0)$ and

$M(r, f) < C(1-r/r_0)^{-\gamma}$ where $C, \gamma > 0$ and $M(r, f)$ is as defined above. Then, as $n \rightarrow \infty$ we have

$$|a_n| = r_0^{-n} \begin{cases} O(1)n^{-\gamma/2}, & (\gamma < 1/2), \\ O(1)(\log n), & (\gamma = 1/2), \\ o(1), & (\gamma > 1/2), \end{cases}$$

where $O(1)$ and $o(1)$ depend on $a, b, \alpha, \beta, \gamma$ and f only.

PROOF OF THEOREM 2.1. From (1.3) and Theorem B we deduce, using the bounds

$|\sinh(s+i\tau)| > \sinh s$, $|\cosh(s+i\tau)| < \cosh s$ and (1.4), that

$$\begin{aligned} \frac{1}{2} (n+\alpha+\beta+1) |a_n| &< (K_{n-1}^{(\alpha+1, \beta+1)} / h_{n-1}^{(\alpha+1, \beta+1)}) (\cosh I_1(r, f') / \sinh^n s) \\ &< (K_{n-1}^{(\alpha+1, \beta+1)} / h_{n-1}^{(\alpha+1, \beta+1)}) (2^n I_1(r, f') / r^n (1-1/r)) \\ |a_n| = r_0^{-n} &\begin{cases} O(1)n^{-\gamma/2}, & (\gamma > 1/2), \\ O(1)(\log n), & (\gamma = 1/2), \\ o(1), & (\gamma < 1/2), \end{cases} \end{aligned}$$

where we have chosen $r = ((n-1)/n)r_0$ and provided that $1-n/(n-1)r_0 > 0$. This completes the proof of Theorem 2.1.

COROLLARY 2.1. Let $f(z) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha, \beta)}(z)$ be mean p -valent in $\text{Int}(E_0)$. Then, as $n \rightarrow \infty$ we have

$$|a_n| = r_0^{-n} \begin{cases} O(1)n^{2p-1/2}, & (p > 1/4), \\ O(1)(\log n), & (p = 1/4), \\ o(1), & (p < 1/4), \end{cases}$$

where $O(1)$ and $o(1)$ depend on a, b, α, β, p and f only. In view of Theorem A, the proof of Corollary 2.1 follows by setting $\gamma = 2p$ in Theorem 2.1.

COROLLARY 2.2. Let $f(z) = \sum_{n=0}^{\infty} a_n p_n^{(\alpha, \beta)}(z)$ be univalent in $\text{Int}(E_o)$. Then as $n \rightarrow \infty$ we have

$$|a_n| = O(1)n^{3/2}r_o^{-n}$$

where $O(1)$ depends on α, b, α, β and f only.

This corollary follows upon setting $p = 1$ in Corollary 2.1.

REMARK. Using the formula (4.21.2) of [1] and the argument used in [2, Remark 2] we see by setting $z = \xi \cosh s_o$ where $|\xi| = |\cos \tau + i \tanh s_o \sin \tau| < 1$ that

$$\begin{aligned} f(\xi \cosh s_o) &= \sum_{n=0}^{\infty} \frac{\Gamma(2n+\alpha+\beta+1)}{n! \Gamma(n+\alpha+\beta+1)} a_n \left(\frac{\cosh s_o}{2}\right)^n \{(\xi - 1/\cosh s_o)^n \\ &\quad + c_1(\xi - 1/\cosh s_o)^{n-1} + \dots + c_n/\cosh^n s_o\} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \hat{a}_n \hat{p}_n^{(\alpha, \beta)}(\xi)$$

where

$$\hat{p}_n^{(\alpha, \beta)}(\xi) = (\xi - 1/\cosh s_o)^n + c_1(\xi - 1/\cosh s_o)^{n-1} + \dots + c_n/\cosh^n s_o$$

and

$$\hat{a}_n = \Gamma(2n+\alpha+\beta+1)a_n \cosh^n s_o / 2^n \Gamma(n+1) \Gamma(n+\alpha+\beta+1).$$

Using this and Stirling's formula and letting $r_o \rightarrow \infty$ we see that Theorem 2.1 and Corollaries 2.1 and 2.2 correspond to analogous results for the unit disk (see Hayman [6]).

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