REMARKS ON QUASILINEAR EVOLUTIONS EQUATIONS

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ABSTRACT. In this paper we study the existence result of classical solutions for the quasilinear equation $u_{tt} - \Delta u - M(\int_{\Omega} |\nabla u|^2 dx) \Delta u_{tt} = f$, with initial data $u(0) = u_0$, $u_t(0) = u_1$ and homogeneous boundary conditions.

KEY WORDS.- Partial differential equation, quasilinear evolution equation, boundary problem.

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1. INTRODUCTION: Let Ω be an open and bounded set of $\mathbb{R}^n$, with smooth boundary Γ . Let's denote by Q the cylinder $Q = \Omega \times]0, T[$ and by Σ its lateral boundary. Our notations and function spaces are standart and follows the same pattern as Lions's book [2].

Ebihara et al [1] was proved that there exist only one classical solution for a semilinear model, given by following initial-boundary value problem

$$u_{tt} - \Delta u - M(\int_{\Omega} |\nabla u|^2 dx) \Delta u_{tt} = f \quad \text{in } Q \quad (1.1)$$

$$u(0) = u_0, \quad u_t(0) = u_1 \quad \text{in } \Omega \quad (1.2)$$

$$u(x, t) = 0 \quad \text{in } \Sigma \quad (1.3)$$

when the following hypotheses hold:

(i) $M(\lambda) \in C^1(0, +\infty)$, and there exist positive constants α, ρ such that the following inequality is valid:

$$M(\lambda) \geq \alpha\sqrt{\lambda} + \rho, \quad \forall \lambda \in [0, +\infty)$$

(ii) There exists a non negative function $\beta(\lambda)$ satisfying:

$$\left| \frac{d}{d\lambda} M(\lambda) \right| \sqrt{\lambda} \leq \beta(\lambda) M(\lambda) \quad \forall \lambda \geq 0$$

(iii) The initial data are such that:

$$\begin{aligned} u_0, u_1 &\in D(A^{(l+1)/2}), \quad l \geq 1 \\ f, \frac{d}{dt}f &\in C(0, T; D(A^{l/2})), \quad l \geq 1 \end{aligned}$$

Where $A = -\Delta$ and for $D(A^s)$ we are denoting the domain of the operator A^s . The main result of this paper is to prove the existence result of classical solutions for system (1.1)-(1.3) when

H1. M is a continuous function such that: $M(\lambda) \geq m_0 > 0$

H2. $f \in C(0, T; D(A^{l/2})), \quad l \geq 2$ and $u_0, u_1 \in D(A^{(l+1)/2}), \quad l \geq 2$

2. THE MAIN RESULT: Let's denote by $w_1, \dots, w_m$ and by $\lambda_1, \dots, \lambda_m$ the m first orthonormal eigen functions and eigen values of the Laplacian respectively. Let's denote by V_m the finite dimensional vector space generated by the first m eigen functions and by P_m the projector operator on V_m , that is:

$$P_m v = \sum_{i=1}^m \left(\int_{\Omega} v(x) w_i(x) dx \right) w_i$$

It is easy to see that $A^s P_m = P_m A^s$ in $D(A^s)$. Moreover we have that

$$\int_{\Omega} |P_m w|^2 dx \leq \int_{\Omega} |w|^2 dx \quad (2.1)$$

Then the approximated problem is defined as follows.

$$\begin{aligned} u_{tt}^{(m)} - \Delta u^{(m)} - M \int_{\Omega} |\nabla u^{(m)}|^2 dx \Delta u_{tt}^{(m)} &= f_m \\ u^{(m)}(0) &= u_0^m, \quad u_t^{(m)}(0) = u_1^m \quad \text{in } \Omega \end{aligned} \quad (2.2)$$

where

$$u^{(m)}(t) = \sum_{i=1}^m g_i^{(m)}(t) w_i, \quad u_0^m = P_m u_0, \quad u_1^m = P_m u_1$$

Before to prove the main result of this paper we will show the following Lemmas:

LEMMA 2.1- Let's suppose that $v, v_t, v_{tt} \in C(0, T; L^2(\Omega))$ and

$$\int_{\Omega} |v_{tt}(x, t)|^2 dx \leq a + b \int_{\Omega} |v(x, t)|^2 dx$$

Then we have:

$$\int_{\Omega} |u(x, t)|^2 dx \leq (a+2b) \int_{\Omega} |u(x, 0)|^2 dx + 4bt^2 \int_{\Omega} |v_t(x, 0)|^2 dx e^{4bt^4}$$

PROOF.- Since

$$u(x, t) = \int_0^t v_t(x, \xi) d\xi + u(x, 0) \quad \text{a. e. in } x$$

we have:

$$|u(x, t)| \leq \sqrt{t} \left(\int_0^t |v_t(x, \xi)|^2 d\xi \right)^{1/2} + |u(x, 0)|$$

From where it follows

$$\int_{\Omega} |u(x, t)|^2 dx \leq 2t \int_0^t \int_{\Omega} |v_t(x, \xi)|^2 dx d\xi + 2 \int_{\Omega} |u(x, 0)|^2 dx$$

Applying the relation above to v_t we have:

$$\int_{\Omega} |v_t(x, t)|^2 dx \leq 2t \int_0^t \int_{\Omega} |v_{tt}(x, \xi)|^2 dx d\xi + 2 \int_{\Omega} |v_t(x, 0)|^2 dx$$

From the two last inequalities we conclude:

$$\begin{aligned} \int_{\Omega} |u(x, t)|^2 dx \leq \\ 2 \int_{\Omega} |u(x, 0)|^2 dx + 4t^2 \int_{\Omega} |v_t(x, 0)|^2 dx + 4t^3 \int_0^t \int_{\Omega} |v_{tt}(x, \xi)|^2 dx d\xi \end{aligned}$$

Finally, from the hypotheses, the last inequality and Gronwall's inequality the result of Lemma 2.1 follows $\square$

LEMMA 2.2.- Let suppose that $w \in C([0, T]; L^2(\Omega))$, then we have that

$$P_m w \rightarrow w \quad \text{strong in } C([0, T]; L^2(\Omega))$$

PROOF.- By the pointwise convergence of $P_m w$ in t , it's sufficient to show that $P_m w$ is a Cauchy sequence in $C([0, T]; L^2(\Omega))$. Let's take $\epsilon > 0$, by the continuity of w we have that there exist $\delta > 0$ such that

$$|t - s| < \delta \Rightarrow \int_{\Omega} |w(x, t) - w(x, s)|^2 dx < \frac{\epsilon}{3} \quad (2.3)$$

By the compactity of $[0, T]$, there exist $s_1, s_2, \dots, s_N$, satisfying

$$[0, T] \subset \bigcup_{i=1}^N]s_i - \delta, s_i + \delta[$$

and from the pointwise convergence of $P_m w$ we conclude that there exists a positive number N such that

$$\int_{\Omega} |P_m w(\cdot, s_i) - P_{\mu} w(\cdot, s_i)|^2 dx < \frac{\epsilon}{3}, \quad \forall m, \mu \geq N, i = 1, \dots, N \quad (2.4)$$

Finally by (2.1), (2.3), (2.4) and the following inequality

$$\begin{aligned} \left(\int_{\Omega} |P_m w(x, t) - P_{\mu} w(x, t)|^2 dx \right)^{1/2} \leq \\ \left(\int_{\Omega} |P_m (w(x, t) - w(x, s_i))|^2 dx \right)^{1/2} + \left(\int_{\Omega} |P_m w(x, s_i) - P_{\mu} w(x, s_i)|^2 dx \right)^{1/2} + \\ + \left(\int_{\Omega} |P_{\mu} (w(x, s_i) - w(x, t))|^2 dx \right)^{1/2} \end{aligned}$$

the result of Lemma 2.2 follows $\square$

THEOREM 2.3.- Let's suppose that H1 and H2 are valid. Then there exists

(1.1), (1.2) and (1.3). Remains to show that u is a classical solution. Let's note that $u^{(m)}$ belongs to $C^2(0, T; D(A^{(l+1)/2}))$ for all $m \in \mathbb{N}$, then in order to prove that $u \in C^2(0, T; C^k(\Omega))$, we will show that $(u_{tt}^{(m)})_{m \in \mathbb{N}}$ is a Cauchy's sequence in $L^\infty(0, T; D(A^{(l+1)/2}))$, for all $l \geq 2$. In fact let $\mu \in \mathbb{N}$, then

$$u_{tt}^{(\mu)} - \Delta u^{(\mu)} - M \int_{\Omega} |\nabla u^{(\mu)}|^2 dx \Delta u_{tt}^{(\mu)} = P_{\mu} f$$

From (2.2) and the above equation we have:

$$(u_{tt}^{(m)} - u_{tt}^{(\mu)}) - \Delta(u^{(m)} - u^{(\mu)}) - M \int_{\Omega} |\nabla u^{(m)}|^2 dx \Delta(u_{tt}^{(m)} - u_{tt}^{(\mu)}) = G_{m\mu}$$

where

$$G_{m\mu} = (M \int_{\Omega} |\nabla u^{(m)}|^2 dx - M \int_{\Omega} |\nabla u^{(\mu)}|^2 dx) \Delta u_{tt}^{(\mu)} + P_{m\mu} f - P_{\mu} f$$

Multiplying the system above by $A^l(u_{tt}^{(m)} - u_{tt}^{(\mu)})$ and integrating in Ω we have

$$m \int_{\Omega} |A^{\frac{l+1}{2}}(u_{tt}^{(m)} - u_{tt}^{(\mu)})|^2 dx \leq$$

$$\int_{\Omega} |A(u^{(m)} - u^{(\mu)}) A^l(u_{tt}^{(m)} - u_{tt}^{(\mu)})| dx + \int_{\Omega} |G_{m\mu} A^l(u_{tt}^{(m)} - u_{tt}^{(\mu)})| dx$$

From which it follows that:

$$\frac{1}{2} m^2 \int_{\Omega} |A^{\frac{l+1}{2}}(u_{tt}^{(m)} - u_{tt}^{(\mu)})|^2 dx \leq \int_{\Omega} |A^{\frac{l+1}{2}}(u^{(m)} - u^{(\mu)})|^2 dx + \int_{\Omega} |A^{\frac{l}{2}} G_{m\mu}|^2 dx$$

From Lemma (3.1) and the last inequality we have

$$\frac{1}{2} m^2 \int_{\Omega} |A^{\frac{l+1}{2}}(u_{tt}^{(m)} - u_{tt}^{(\mu)})|^2 dx \leq$$

$$(\int_{\Omega} |A^{\frac{l}{2}} G_{m\mu}|^2 dx + 2 \int_{\Omega} |A^{\frac{l+1}{2}}(u_0^{(m)} - u_0^{(\mu)})|^2 dx + 4t^2 \int_{\Omega} |A^{\frac{l+1}{2}}(u_1^{(m)} - u_1^{(\mu)})|^2 dx) \exp(\frac{8}{m^2} t^4)$$

Finally from Lemma 2.2 and since $u_0, u_1 \in D(A^{(l+1)/2})$ we have that

$$A^{l/2} G_{m\mu} \rightarrow 0 \text{ as } m, \mu \rightarrow +\infty \text{ strongly in } C([0, T]; L^2(\Omega))$$

Then we have that $(u_{tt}^{(m)})$ a Cauchy sequence in $L^\infty(0, T; D(A^{(l+1)/2}))$ and the proof is now complete $\square$

REMARK 2.4.- UNIQUENESS: If M is locally Lipschitz, then we have uniqueness. In fact, let u and v be two solutions, putting $w = u - v$ we have

$$w_{tt} - \Delta w - M \int_{\Omega} |\nabla u|^2 dx \Delta w_{tt} = (M \int_{\Omega} |\nabla u|^2 dx - M \int_{\Omega} |\nabla v|^2 dx) \Delta w_{tt}$$

Multiplying by Δw_{tt} applying H1 and the Lipschitz condition on M we have that there exists a positive constant c_1 such that:

$$m \int_{\Omega} |\Delta w_{tt}|^2 dx \leq \int_{\Omega} |\Delta w \Delta w_{tt}| dx + c_1 (\int_{\Omega} |\Delta w|^2 dx)^{1/2} (\int_{\Omega} |\Delta w_{tt}|^2 dx)^{1/2}$$

only one classic solution of system (1.1), (1.2) and (1.3)

PROOF.— Since $DA^{(l+1)/2} \in H^{l+1}(\Omega) \subset C^k(\bar{\Omega})$ if $l+1 > \frac{n}{2} + k$, it's sufficient to show that there exists a solution of system (1.1), (1.2) and (1.3) satisfying $u \in C^2([0, T]; DA^{(l+1)/2})$. In order to prove it let's multiply (2.2) by $A^l u_{tt}^{(m)}$ and integrating in Ω we have:

$$\int_{\Omega} |A^{\frac{l}{2}} u_{tt}^{(m)}|^2 dx + M \int_{\Omega} |\nabla u^{(m)}|^2 dx \int_{\Omega} |A^{\frac{l+1}{2}} u_{tt}^{(m)}|^2 dx = \\ - \int_{\Omega} A u^{(m)} A^l u_{tt}^{(m)} dx + \int_{\Omega} f_m A^l u_{tt}^{(m)} dx$$

By H1 and H2 the last equality becomes:

$$m_0 \int_{\Omega} |A^{\frac{l+1}{2}} u_{tt}^{(m)}|^2 dx \leq \int_{\Omega} |A^{\frac{l+1}{2}} u^{(m)} A^{\frac{l+1}{2}} u_{tt}^{(m)}| dx + \int_{\Omega} |A^{\frac{l}{2}} f_m| A^{\frac{l}{2}} u_{tt}^{(m)}| dx$$

from where it follows that:

$$\frac{1}{2} m_0^2 \int_{\Omega} |A^{\frac{l+1}{2}} u_{tt}^{(m)}|^2 dx \leq \frac{1}{\lambda_1^2} \int_{\Omega} |A^{\frac{l}{2}} f_m|^2 dx + \int_{\Omega} |A^{\frac{l+1}{2}} u^{(m)}|^2 dx$$

By Lemma 2.1 and the above inequality we obtain:

$$\frac{1}{2} m_0^2 \int_{\Omega} |A^{\frac{l+1}{2}} u_{tt}^{(m)}(x, t)|^2 dx \leq \\ (\int_{\Omega} |A^{\frac{l}{2}} f_m|^2 dx + 2 \int_{\Omega} |A^{\frac{l+1}{2}} u_0^m|^2 dx + 4t^2 \int_{\Omega} |A^{\frac{l+1}{2}} u_1^m|^2 dx) \exp(\frac{8}{m_0^2} t^4) \quad (2.5)$$

From (2.5) and since:

$$\int_{\Omega} |A^{\frac{l+1}{2}} u_t^{(m)}(x, t)|^2 dx \leq 2t \int_{\Omega} |A^{\frac{l+1}{2}} u_{tt}^{(m)}(x, t)|^2 dx + 2 \int_{\Omega} |A^{\frac{l+1}{2}} u_1^m|^2 dx \\ \int_{\Omega} |A^{\frac{l+1}{2}} u^{(m)}(x, t)|^2 dx \leq 2t \int_{\Omega} |A^{\frac{l+1}{2}} u_t^{(m)}(x, t)|^2 dx + 2 \int_{\Omega} |A^{\frac{l+1}{2}} u_0^m|^2 dx$$

we conclude that there exists a subsequence of $\{u^{(m)}\}_{m \in \mathbb{N}}$, which we still denoting of the same way and a function $u \in L^{\infty}(0, T; DA^{(l+1)/2})$, satisfying

$$u^{(m)} \rightarrow u \text{ weak star in } L^{\infty}(0, T; DA^{(l+1)/2}) \text{ as } m \rightarrow \infty \\ u_t^{(m)} \rightarrow u_t \text{ weak star in } L^{\infty}(0, T; DA^{(l+1)/2}) \text{ as } m \rightarrow \infty \\ u_{tt}^{(m)} \rightarrow u_{tt} \text{ weak star in } L^{\infty}(0, T; DA^{(l+1)/2}) \text{ as } m \rightarrow \infty$$

From the last convergences and the Lions-Aubin's theorem (see Lions's [2], theorem 5.1, chap 1) we conclude in particular that:

$$u^{(m)} \rightarrow u \text{ strongly in } C([0, T]; H_0^1(\Omega)) \text{ as } m \rightarrow \infty$$

By standard methods we can prove that u is a strong solution of system

from where it follows that there exists c_2 such that:

$$\int_{\Omega} |\Delta w_{tt}|^2 dx \leq c_2 \int_{\Omega} |\Delta w|^2 dx$$

By Lemma 2.1, since $w(x,0) = w_t(x,0) = 0$, we obtain that $\Delta w = 0$, and from this it follows that $w = 0$, that is $u = v$ $\square$

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