AN INVERSE EIGENVALUE PROBLEM FOR AN ARBITRARY MULTIPLY CONNECTED BOUNDED REGION IN $\mathbb{R}^2$

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ABSTRACT. The basic problem is to determine the geometry of an arbitrary multiply connected bounded region in $\mathbb{R}^2$ together with the mixed boundary conditions, from the complete knowledge of the eigenvalues \( \{\lambda_j\}_{j=1}^\infty \) for the Laplace operator, using the asymptotic expansion of the spectral function \( \theta(t) = \sum_{j=1}^\infty \exp(-t\lambda_j) \) as $t \to 0$.

KEY WORDS AND PHRASES. Inverse problem, Laplace’s operator, eigenvalue problem, spectral function.

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1. INTRODUCTION.

The underlying problem is to deduce the precise shape of a membrane from the complete knowledge of the eigenvalues \( \{\lambda_j\}_{j=1}^\infty \) for the Laplace operator $\Delta = \sum_{i=1}^2 \left( \frac{\partial}{\partial x_i} \right)^2$ in the $x^1x^2$-plane.

Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected bounded domain with a smooth boundary $\partial \Omega$. Consider the Neumann/Dirichlet problem

\[ (\Delta + \lambda)u = 0 \quad \text{in} \quad \Omega, \]
\[ \frac{\partial u}{\partial n} = 0 \quad \text{or} \quad u = 0 \quad \text{on} \quad \partial \Omega, \]

where $\frac{\partial}{\partial n}$ denotes differentiation along the inward pointing normal to $\partial \Omega$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Denote its eigenvalues, counted according to multiplicity, by

\[ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots \to \infty \quad \text{as} \quad j \to \infty. \]

The problem of determining the geometry of $\Omega$ has been investigated by Pleijel [1], Kac [2], McKean and Singer [3], Stewartson and Waechter [4], Smith [5], Sleeman and Zayed [6,7], Gottlieb [8], Greiner [9], Zayed [10-13] and the references given there, using the asymptotic expansion of the trace function

\[ \theta(t) = tr[\exp(-t\Delta)] = \sum_{j=1}^\infty \exp(-t\lambda_j) \quad \text{as} \quad t \to 0. \]

It has been shown that, in the case of Neumann boundary conditions (N.b.c.):
\[ \theta(t) = \frac{\Omega}{4\pi t} + \frac{1}{8(\pi t)^{3/2}} \left( \sum_{i=1}^{k} L_i - \sum_{i=k+1}^{m} L_i \right) + \frac{7}{256} \left( \frac{t}{\pi} \right)^{1/2} \int_{\partial \Omega} k^2(\sigma) d\sigma + o(t) \quad \text{as} \quad t \to 0, \] (1.5)

while, in the case of Dirichlet boundary conditions (D.b.c.):

\[ \theta(t) = \frac{\Omega}{4\pi t} + \frac{1}{8(\pi t)^{3/2}} + \frac{1}{256} \left( \frac{t}{\pi} \right)^{1/2} \int_{\partial \Omega} k^2(\sigma) d\sigma + o(t) \quad \text{as} \quad t \to 0, \] (1.6)

In these formulae, \( |\Omega| \) is the area of \( \Omega \), \( |\partial \Omega| \) is the total length of \( \partial \Omega \) and \( k(\sigma) \) is the curvature of \( \partial \Omega \). The constant term \( a_0 \) has geometric significance, e.g., if \( \Omega \) is smooth and convex, then \( a_0 = \frac{1}{6} \) and if \( \Omega \) is permitted to have a finite number of smooth convex holes "H", then \( a_0 = \frac{1}{6}(1 - H) \).

The object of this paper is to discuss the following more general inverse problem: Let \( \Omega \) be an arbitrary multiply connected bounded region in \( \mathbb{R}^2 \) which is surrounded internally by simply connected bounded domains \( \Omega_i \), \( i = 1, \ldots, m-1 \) and externally by a simply connected bounded domain \( \Omega_m \) with a smooth boundary \( \partial \Omega_m \). Suppose that the eigenvalues (1.3) are given for the eigenvalue equation

\[ (\Lambda_\nu + \lambda)u = 0 \quad \text{in} \quad \Omega, \] (1.7)

together with one of the following mixed boundary conditions:

\[ \frac{\partial u}{\partial n_i} = 0 \quad \text{on} \quad \partial \Omega_i, \quad i = 1, \ldots, k \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial \Omega_i, \quad i = k+1, \ldots, m, \] (1.8)

\[ u = 0 \quad \text{on} \quad \partial \Omega_i, \quad i = 1, \ldots, k \quad \text{and} \quad \frac{\partial u}{\partial n_i} = 0 \quad \text{on} \quad \partial \Omega_i, \quad i = k+1, \ldots, m, \] (1.9)

where \( \frac{\partial}{\partial n_i} \) denote differentiations along the inward pointing normals to the boundaries \( \partial \Omega_i, i = 1, \ldots, m \), respectively.

The basic problem is to determine the geometry of \( \Omega \) from the asymptotic expansion of the spectral function (1.4) for small positive \( t \).

Note that problems (1.7)-(1.9) have been investigated recently by Zayed [11] in the special case where \( \Omega \) is an arbitrary doubly connected bounded region (i.e., \( m = 2 \)).

2. STATEMENT OF OUR RESULTS.

Suppose that the boundaries \( \partial \Omega_i, i = 1, \ldots, m \) are given locally by the equations \( x^n - y^n(\sigma_i), n = 1, 2 \) in which \( \sigma_i, i = 1, \ldots, m \) are the arc-lengths of the counterclockwise oriented boundaries \( \partial \Omega_i \) and \( y^n(\sigma_i) \in C^n(\partial \Omega_i) \). Let \( L_i \) and \( k_i(\sigma_i) \) be the lengths and the curvatures of \( \partial \Omega_i, i = 1, \ldots, m \) respectively. Then, the results of our main problem (1.7)-(1.9) can be summarized in the following cases:

CASE 1. (N.b.c. on \( \partial \Omega_i, i = 1, \ldots, k \) and D.b.c. on \( \partial \Omega_i, i = k+1, \ldots, m \))

\[ \theta(t) = \frac{|\Omega|}{4\pi t} + \frac{1}{8(\pi t)^{3/2}} \left( \sum_{i=1}^{k} L_i - \sum_{i=k+1}^{m} L_i \right) + \frac{1}{6} (2 - m) \]

\[ + \frac{1}{256} \left( \frac{t}{\pi} \right)^{1/2} \left( \sum_{i=1}^{k} \int_{\partial \Omega_i} k_i^2(\sigma_i) d\sigma_i + \sum_{i=k+1}^{m} \int_{\partial \Omega_i} k_i^2(\sigma_i) d\sigma_i \right) \]

\[ + o(t) \quad \text{as} \quad t \to 0. \] (2.1)

CASE 2. (D.b.c. on \( \partial \Omega_i, i = 1, \ldots, k \) and N.b.c. on \( \partial \Omega_i, i = k+1, \ldots, m \))

In this case the asymptotic expansion of \( \theta(t) \) as \( t \to 0 \) has the same form (2.1) with the interchanges \( \partial \Omega_i, i = 1, \ldots, k \leftrightarrow \partial \Omega_i, i = k+1, \ldots, m \).
With reference to formulae (1.4), (1.5) and to articles [6], [11], [12] the asymptotic expansion (2.1) may be interpreted as follows:

(i) Ω is an arbitrary multiply connected bounded region in $\mathbb{R}^2$ and we have the mixed boundary conditions (1.8) or (1.9) as indicated in the specifications of the two respective cases.

(ii) For the first four terms, Ω is an arbitrary multiply connected bounded region in $\mathbb{R}^2$ of area $|\Omega|$. In case 1, it has $H = (m-1)$ holes, the boundaries $\partial \Omega_i$, $i = 1, \ldots, k$ are of lengths $\sum_{i=1}^{k} L_i$ and of curvatures $k_i(\sigma_i)$, $i = 1, \ldots, k$ together with Neumann boundary conditions, while the boundaries $\partial \Omega_i$, $i = k+1, \ldots, m$ are of lengths $\sum_{i=k+1}^{m} L_i$ and of curvatures $k_i(\sigma_i)$, $i = k+1, \ldots, m$ together with Dirichlet boundary conditions, provided $H$ is an integer.

We close this section with the following remarks:

**REMARK 2.1.** On setting $k = 0$ in formula (2.1) with the usual definition that $\sum_{i=1}^{0}$ is zero, we obtain the results of Dirichlet boundary conditions on $\partial \Omega_i$, $i = 1, \ldots, m$.

**REMARK 2.2.** On setting $k = m$ in formula (2.1) with the usual definition that $\sum_{i=m+1}^{m}$ is zero, we obtain the results of Neumann boundary conditions on $\partial \Omega_i$, $i = 1, \ldots, m$.

### 3. FORMULATION OF THE MATHEMATICAL PROBLEM

It is easy to show that the spectral function (1.4) associated with problems (1.7)-(1.9) is given by

$$\theta(t) = \int_{\Omega} G\left(\frac{x_1}{x_2}, \frac{x_1}{x_2}; \frac{t}{4}\right) dx,$$

where $G\left(\frac{x_1}{x_2}; \frac{t}{4}\right)$ is Green's function for the heat equation

$$\left(\Delta_2 - \frac{\partial}{\partial t}\right) u = 0,$$

subject to the mixed boundary conditions (1.8) or (1.9) and the initial condition

$$\lim_{t \to 0} G\left(\frac{x_1}{x_2}; \frac{t}{4}\right) = \delta\left(\frac{x_1}{x_2} - 1\right),$$

where $\delta\left(\frac{x_1}{x_2} - 1\right)$ is the Dirac delta function located at the source point $x_1 = x_2$. Let us write

$$G\left(\frac{x_1}{x_2}, \frac{x_1}{x_2}; \frac{t}{4}\right) = G_0\left(\frac{x_1}{x_2}, \frac{x_1}{x_2}; \frac{t}{4}\right) + \chi\left(\frac{x_1}{x_2}, \frac{x_1}{x_2}; \frac{t}{4}\right),$$

where

$$G_0\left(\frac{x_1}{x_2}, \frac{x_1}{x_2}; \frac{t}{4}\right) = (4\pi t)^{-\frac{3}{2}} \exp\left\{-\frac{\left|\frac{x_1}{x_2} - 1\right|^2}{4t}\right\},$$

is the "fundamental solution" of the heat equation (3.2), while $\chi\left(\frac{x_1}{x_2}, \frac{x_1}{x_2}; \frac{t}{4}\right)$ is the "regular solution" chosen so that $G\left(\frac{x_1}{x_2}, \frac{x_1}{x_2}; \frac{t}{4}\right)$ satisfies the mixed boundary conditions (1.8) or (1.9).

On setting $x_1 = x_2 = x$ we find that
where

\[
K(t) = \int_0^T \chi(x, x; t) \, dx.
\]

The problem now is to determine the asymptotic expansion of \(K(t)\) for small positive \(t\). In what follows we shall use Laplace transforms with respect to \(t\), and use \(s^2\) as the Laplace transform parameter; thus we define

\[
\bar{G}(x_1, x_2; s^2) = \int_0^\infty e^{-s^2} G(x_1, x_2; t^2) \, dt.
\]

An application of the Laplace transform to the heat equation (3.2) shows that \(\bar{G}(x_1, x_2; s^2)\) satisfies the membrane equation

\[
(A_2 - s^2)\bar{G}(x_1, x_2; s^2) = -\delta(x_1 - x_2) \quad \text{in} \quad \Omega,
\]

together with the mixed boundary conditions (1.8) or (1.9).

The asymptotic expansion of \(K(t)\) for small positive \(t\), may then be deduced directly from the asymptotic expansion of \(\bar{K}(s^2)\) for large positive \(s\), where

\[
\bar{K}(s^2) = \int_\Omega \chi(x_1, x_2; s^2) \, dx.
\]

4. CONSTRUCTION OF GREEN'S FUNCTION.

It is well known [6] that the membrane equation (3.9) has the fundamental solution

\[
\bar{G}_0(x_1, x_2; s^2) = \frac{1}{2\pi} K_0(s r_{x_1 x_2})
\]

where \(r_{x_1 x_2} = |x_1 - x_2|\) is the distance between the points \(x_1 = (x_1^1, x_1^2)\) and \(x_2 = (x_2^1, x_2^2)\) of the region \(\Omega\) while \(K_0\) is the modified Bessel function of the second kind and of zero order. The existence of this solution enables us to construct integral equations for \(\bar{G}(x_1, x_2; s^2)\) satisfying the mixed boundary conditions (1.8) or (1.9). Therefore, Green's theorem gives:

CASE 1. (N.b.c. on \(\partial \Omega, \ i = 1, \ldots, k\) and D.b.c. on \(\partial \Omega, \ i = k + 1, \ldots, m\))

\[
\bar{G}(x_1, x_2; s^2) = \frac{1}{2\pi} K_0(s r_{x_1 x_2}) + \frac{1}{\pi} \sum_{i=1}^k \int_{\partial \Omega_i} \bar{G}(x_1, y; s^2) \frac{\partial}{\partial n_y} K_0(s r_{x_1 y}) \, dy + \frac{1}{\pi} \sum_{i=k+1}^m \int_{\partial \Omega_i} \frac{\partial}{\partial n_y} \bar{G}(x_1, y; s^2) K_0(s r_{x_1 y}) \, dy.
\]

CASE 2. (D.b.c. on \(\partial \Omega, \ i = 1, \ldots, k\) and N.b.c. on \(\partial \Omega, \ i = k + 1, \ldots, m\))

In this case Green’s function \(\bar{G}(x_1, x_2; s^2)\) has the same form (4.2) with the interchanges \(\partial \Omega_i, \ i = 1, \ldots, k \leftrightarrow \partial \Omega_i, \ i = k + 1, \ldots, m\).
On applying the iteration method (see [11], [12]) to the integral equation (4.2), we obtain Green’s function \( \mathcal{G}(x_i, x_2; s^2) \) which has the regular part:

\[
\mathcal{G}(x_i, x_2; s^2) = \frac{1}{2\pi i} \sum_{\alpha_i} \int K_0(sr_{\alpha_i}) \frac{\partial}{\partial n_{\alpha_i}} K_0(sr_{\alpha_i}) \, dy \\
+ \frac{1}{2\pi i} \sum_{\alpha_i} \int K_0(sr_{\alpha_i}) M_i(y, y') \frac{\partial}{\partial n_{\alpha_i}} K_0(sr_{\alpha_i}) \, dy \, dy' \\
+ \frac{1}{2\pi i} \sum_{\alpha_i} \int K_0(sr_{\alpha_i}) L_i(y, y') K_0(sr_{\alpha_i}) \, dy \, dy' \\
+ \frac{1}{2\pi i} \sum_{\alpha_i} \int \left[ \sum_{\alpha_i} \int K_0(sr_{\alpha_i}) L_i(y, y') \frac{\partial}{\partial n_{\alpha_i}} K_0(sr_{\alpha_i}) \, dy \, dy' \right] K_0(sr_{\alpha_i}) \, dy',
\]

(4.3)

where

\[
M_i(y, y') = \sum_{\alpha_i} K^{(\nu)}(y, y'), \tag{4.4}
\]

\[
M_i^T(y, y') = \sum_{\alpha_i} K^{(\nu)}(y', y), \tag{4.5}
\]

\[
L_i(y, y') = \sum_{\alpha_i} K^{(\nu)}(y', y), \tag{4.6}
\]

\[
L_i^T(y, y') = \sum_{\alpha_i} K^{(\nu)}(y', y), \tag{4.7}
\]

\[
K_i(y', y) = \frac{1}{\pi} \frac{\partial}{\partial n_y} K_0(sr_{yy}), \tag{4.8}
\]

\[
* K_i(y', y) = \frac{1}{\pi} \frac{\partial}{\partial n_{y'}} K_0(sr_{yy}), \tag{4.9}
\]

\[
K(y', y) = \frac{1}{\pi} K_0(sr_{yy}), \tag{4.10}
\]

and

\[
* K(y', y) = \frac{1}{\pi} \frac{\partial^2}{\partial n_{y'} \partial n_{y'}} K_0(sr_{yy}). \tag{4.11}
\]

In the same way, we can show that in case 2 Green’s function \( \mathcal{G}(x_i, x_2; s^2) \) has a regular part of the same form (4.3) with the interchanges \( \partial \Omega_i, \ i = 1, \ldots, k \leftrightarrow \partial \Omega_i, \ i = k + 1, \ldots, m \).
On the basis of (4.3) the function \( \chi(x_1, x_2; s^2) \) will be estimated for large values of \( s \). The case when \( x_1 \) and \( x_2 \) lie in the neighborhoods of \( \partial \Omega_i, i = 1, \ldots, m \) is particularly interesting. For this case, we need to use the following coordinates.

5. COORDINATES IN THE NEIGHBORHOODS OF \( \partial \Omega_i, i = 1, \ldots, m \).

Let \( n_i, i = 1, \ldots, m \) be the minimum distances from a point \( x = (x_1, x_2) \) of the region \( \Omega \) to the boundaries \( \partial \Omega_i, i = 1, \ldots, m \) respectively. Let \( n_i(\sigma_i), i = 1, \ldots, m \) denote the inward drawn unit normals to \( \partial \Omega_i, i = 1, \ldots, m \) respectively. We note that the coordinates in the neighborhood of \( \partial \Omega_i, i = k + 1, \ldots, m \) and its diagrams (see [11]) are in the same form as in section 5.1 of [11] with the interchanges \( \sigma_2 \leftrightarrow \sigma_1, n_2 \leftrightarrow n_1, h_2 \leftrightarrow h_1, l_2 \leftrightarrow l_1, D(l_2) \leftrightarrow D(l_1) \) and \( \delta_2 \leftrightarrow \delta_1, i = k + 1, \ldots, m \). Thus, we have the same formulæ (5.1.1)-(5.1.5) of section 5.1 in [11] with the interchanges \( n_2 \leftrightarrow n_1, n_2(\sigma_2) \leftrightarrow n_1(\sigma_1), t_2(\sigma_2) \leftrightarrow t_1(\sigma_1), k_2(\sigma_2) \leftrightarrow k_1(\sigma_1), i = k + 1, \ldots, m \).

Similarly, the coordinates in the neighborhood of \( \partial \Omega_i, i = 1, \ldots, k \) and its diagrams (see [11]) are similar to those obtained in section 5.2 of [11] with the interchanges \( \sigma_1 \leftrightarrow \sigma_2, n_1 \leftrightarrow n_2, h_1 \leftrightarrow h_2, l_1 \leftrightarrow l_2, D(l_1) \leftrightarrow D(l_2) \) and \( \delta_1 \leftrightarrow \delta_2, i = 1, \ldots, k \). Thus, we have the same formulæ (5.2.1)-(5.2.5) of section 5.2 in [11] with the interchanges \( n_1 \leftrightarrow n_2, n_1(\sigma_1) \leftrightarrow n_2(\sigma_2), t_1(\sigma_1) \leftrightarrow t_2(\sigma_2) \) and \( k_1(\sigma_1) \leftrightarrow k_2(\sigma_2), i = 1, \ldots, k \).

6. SOME LOCAL EXPANSIONS.

It now follows that the local expansions of the functions
\[
K_i\left(\frac{sr_{xy}}{2}\right), \quad \frac{\partial}{\partial n_{xy}}K_i\left(\frac{sr_{xy}}{2}\right), \quad i = 1, \ldots, m
\]
when the distance between \( x \) and \( y \) is small, are very similar to those obtained in section 6 of [11]. Consequently, for \( i = 1, \ldots, k, k + 1, \ldots, m \), the local behavior of the following kernels:
\[
K_i\left(y', y\right), \quad K_i\left(y', y\right)
\]
when the distance between \( y \) and \( y' \) is small, follows directly from the knowledge of the local expansions of (6.1).

**DEFINITION 1.** Let \( \xi_i \) and \( \xi_j \) be points in the upper half-plane \( \xi^2 > 0 \), then we define
\[
\rho_{ij} = \sqrt{(\xi_i^1 - \xi_j^1)^2 + (\xi_i^2 + \xi_j^2)^2}.
\]
An \( \epsilon^{\rho_{ij}}(\xi_i, \xi_j, s^2) \)-function is defined for points \( \xi_i \) and \( \xi_j \) belong to sufficiently small domains \( D(I_i) \) except when \( \xi_i = \xi_j \in I_i, i = 1, \ldots, m \) and \( \lambda \) is called the degree of this function. For every positive integer \( \Lambda \) it has the local expansion (see [11]):
\[
e^A\left(\xi_1, \xi_2; s\right) = \sum f(h) \left(\xi_1, \xi_2\right) \left[\frac{\partial}{\partial h}\right] (s) + R^A\left(\xi_1, \xi_2; s\right), \tag{6.5}
\]

where \(\sum^\prime\) denotes a sum of a finite number of terms in which \(f(h)\) is an infinitely differentiable function. In this expansion, \(P_1, P_2, l, m\) are integers, where \(P_1 \geq 0, P_2 \geq 0, l \geq 0, \lambda = \min(P_1 + P_2 - q), q = l + m\) and the minimum is taken over all terms which occur in the summation \(\sum^\prime\). The remainder \(R^A\left(\xi_1, \xi_2; s\right)\) has continuous derivatives of order \(d \leq \Lambda\) satisfying

\[
D^A R^A\left(\xi_1, \xi_2; s\right) = o(s^{-\Lambda + 1/2}) \quad \text{as} \quad s \to \infty, \tag{6.6}
\]

where \(A\) is a positive constant.

Thus, using methods similar to those obtained in section 7 of [11], we can show that the functions (6.1) are \(e^\lambda\)-functions with degrees \(\lambda = 0, -1\) respectively. Consequently, the functions (6.2) are \(e^\lambda\)-functions with degrees \(\lambda = 0, -1\), while the functions (6.3) are \(e^\lambda\)-functions with degrees \(\lambda = 0, 1\) respectively.

**DEFINITION 2.** If \(x_1\) and \(x_2\) are points in large domains \(\Omega + \partial \Omega_i, i = 1, \ldots, k, k + 1, \ldots, m\), then we define

\[
\hat{r}_{12} = \min_{y \in \partial \Omega_i} \left( r_{x_1} + r_{x_2} \right) \quad \text{if} \quad y \in \partial \Omega_i, \quad i = 1, \ldots, k,
\]

and

\[
\hat{R}_{12} = \min_{y \in \partial \Omega_i} \left( r_{x_1} + r_{x_2} \right) \quad \text{if} \quad y \in \partial \Omega_i, \quad i = k + 1, \ldots, m.
\]

An \(E^A\left(x_1, x_2; s\right)\)-function is defined and infinitely differentiable with respect to \(x_1\) and \(x_2\) when these points belong to large domains \(\Omega + \partial \Omega_i\) except when \(x_1 = x_2 \in \partial \Omega_i, i = 1, \ldots, m\). Thus, the \(E^\lambda\)-function has a similar local expansion of the \(e^\lambda\)-function (see [6], [11]).

By the help of section 8 in [11], it is easily seen that formula (4.3) is an \(E^A\left(x_1, x_2; s\right)\)-function and consequently

\[
\overline{G}\left(x_1, x_2; s^2\right) = \sum_{i=1} A \left[ 1 + \log s \hat{r}_{12} \right] e^{-A \hat{r}_{12}}.
\]

\[
= \sum_{i=1}^m \sum_{i=1}^m \left[ 1 + \log s \hat{R}_{12} \right] e^{-A \hat{R}_{12}}, \tag{6.7}
\]

which is valid for \(s \to \infty\), where \(A_i, i = 1, \ldots, m\) are positive constants.

Formula (6.7) shows \(\overline{G}\left(x_1, x_2; s^2\right)\) is exponentially small for \(s \to \infty\).

**7. THE ASYMPTOTIC BEHAVIOR OF \(\overline{\chi}\left(x_1, x_2; s^2\right)\).**

With reference to sections 7 and 9 in [11], if the \(e^\lambda\)-expansions of the functions (6.1)-(6.3) are introduced into (4.3) and if we use formulae similar to (7.4) and (7.10) of section 7 in [11], we obtain the following local behavior of \(\overline{\chi}\left(x_1, x_2; s^2\right)\) as \(s \to \infty\) which is valid when \(\hat{r}_{12}\) and \(\hat{R}_{12}\) are small:

\[
\overline{\chi}\left(x_1, x_2; s^2\right) = \sum_{i=1}^m \overline{\chi}_i\left(x_1, x_2; s^2\right), \tag{7.1}
\]
where, if \( x_1 \) and \( x_2 \) belong to sufficiently small domains \( D(l_i), i = 1, \ldots, k, k + 1, \ldots, m, \) then

\[
\overline{\chi}_i(x_1, x_2; s^2) = -\frac{1}{2\pi} K_0(s\hat{r}_{12}) + O\{s^{-1}\exp(-A_1s\hat{r}_{12})\}.
\]  

(7.2)

When \( \hat{r}_{12} \geq \delta_i > 0, i = 1, \ldots, k \) and \( \hat{R}_{12} \geq \delta_i > 0, i = k + 1, \ldots, m \) the function \( \overline{\chi}_i(x_1, x_2; s^2) \) is of order \( O\{\exp(-cs)\} \) as \( s \to \infty, c > 0 \). Thus, since \( \lim_{s \to 0} \overline{\chi}_i(x_1, x_2; s^2) = 0 \), we deduce for \( s \to \infty \) that

\[
\overline{\chi}_i(x_1, x_2; s^2) = -\frac{1}{2\pi} K_0(s\hat{r}_{12}) + O\{s^{-1}\exp(-A_1s\hat{r}_{12})\}.
\]  

(7.3)

while, if \( x_1 \) and \( x_2 \) belong to large domains \( \Omega + \partial \Omega_i, i = 1, \ldots, k \), we deduce for \( s \to \infty \) that

\[
\overline{\chi}_i(x_1, x_2; s^2) = -\frac{1}{2\pi} K_0(s\hat{r}_{12}) + O\{s^{-1}\exp(-A_1s\hat{r}_{12})\}.
\]  

(7.4)

8. CONSTRUCTION OF OUR RESULTS.

Since for \( \xi_i \geq h_i > 0, i = 1, \ldots, k, k + 1, \ldots, m, \) the functions \( \overline{\chi}_i(x_1, x_2; s^2) \) are of order \( O\{\exp(-2\xi_i h_i)\} \), the integral of the function \( \overline{\chi}_i(x_1, x_2; s^2) \) over the region \( \Omega \) can be approximated in the following way (see (3.10)):

\[
\overline{K}(s^2) = \sum_{i=1}^{m} \int_{\xi_i = h_i}^{l_i} \int_{\eta_i = 0}^{l_i} \overline{\chi}_i(x_1, x_2; s^2) \{1 - k_i(\xi_i; \eta_i)\} d\xi_i d\eta_i
\]

\[
- \sum_{i=1}^{m} \int_{\xi_i = -\infty}^{h_i} \int_{\eta_i = 0}^{l_i} \overline{\chi}_i(x_1, x_2; s^2) \{1 + k_i(\xi_i; \eta_i)\} d\xi_i d\eta_i
\]

\[
+ \sum_{i=1}^{m} O\{\exp(-2\xi_i h_i)\} \quad \text{as} \quad s \to \infty.
\]  

(8.1)

If the \( e^\lambda \)-expansions of \( \overline{\chi}_i(x_1, x_2; s^2), i = 1, \ldots, k, k + 1, \ldots, m, \) are introduced into (8.1), one obtains an asymptotic series of the form:

\[
\overline{K}(s^2) = \sum_{\alpha=1}^{l} a_\alpha s^{\alpha} + O\{s^{\alpha-1}\} \quad \text{as} \quad s \to \infty,
\]  

(8.2)

where the coefficients \( a_{\alpha} \) are calculated from the \( e^\lambda \)-expansions by the help of formula (10.3) of section 10 in [11].

Now, the first three coefficients \( a_1, a_2, a_3 \) take the forms:
\[
a_1 = \frac{1}{8} \left( \sum_{i=1}^{4} L_i - \sum_{i=1}^{n} L_i \right),
\]
\[
a_2 = \frac{1}{6} (2 - m),
\]
\[
a_3 = \frac{1}{512} \left[ 7 \sum_{i=1}^{4} \int_{\alpha_i} k_i^2(\alpha) d\alpha_i + \sum_{i=1}^{n} \int_{\alpha_i} k_i^2(\alpha) d\alpha_i \right].
\] (8.3)

On inserting (8.3) into (8.2) and inverting Laplace transforms and using (3.6) we arrive at our result (2.1).

REFERENCES


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