A COMMON FIXED POINT THEOREM FOR TWO SEQUENCES OF SELF-MAPPINGS

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ABSTRACT. In this paper a common fixed point theorem for two sequences of self-mappings from a complete metric space M to M is proved. Our theorem is a generalization of Hadzic's fixed point theorem[1].

KEY WORDS AND PHRASES. A common fixed point, self-mappings and complete metric spaces.

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1. INTRODUCTION.

Banach's fixed point theorem has been generalized by many authors. Among such investigations there are several, interesting and important studies[2]. Particularly, K. Iseki[3] proved a fixed point theorem of a sequence of self-mappings from a complete metric space M to M . We are interested in fixed point theorems of a sequence of self-mappings since they pertain to the problem of finding an equilibrium point of a difference equation $x_{n+1} = f(n, x_n)$ ($n = 1, 2, \dots$).

Recently O. Hadzic proved the existence of a common fixed point for the sequence of self-mappings $\{A_j\}(j = 1, 2, \dots)$, S and T where A_j commutes with S and T . His result is as follows:

THEOREM 1. Let (M, d) be a complete metric space, $S, T: M \rightarrow M$ be continuous, $A_j: M \rightarrow SM \cap TM$ ($j = 1, 2, \dots$) so that A_j commutes with S and T and for every i, j ($i = j, i, j = 1, 2, \dots$) and every $x, y \in M$:

$$d(A_i x, A_j y) \leq q d(Sx, Ty), \quad 0 < q < 1 \quad (1.1)$$

Using Theorem 1, he gave a generalization of Gohde's fixed point theorem and extended Krasnosel'ski's fixed point theorem.

In this paper we shall present a generalization of Hadzic's fixed point theorem.

2. MAIN THEOREMS.

Let N denote the set of all positive integers. In this section we shall prove the following theorem.

THEOREM A. Let (M, d) be a complete metric space and let $\{A_p\}, \{B_q\} (p, q = 1, 2, \dots)$, be two sequences of mappings from M to M .

Suppose that the following conditions are satisfied; for all $m, n \in N$ and all $x, y \in M$,

(a) there exists a constant k ($0 < k < 1$) such that

$$d(A_{2n-1}x, A_{2n}y) \leq kd(B_{2n-1}x, B_{2n}y),$$

$$d(A_{2n}x, A_{2m+1}y) \leq kd(B_{2n}x, B_{2m+1}y), \text{ for all } m \geq n \geq 1,$$

(b) $A_{2n}B_{2m} = B_{2m}A_{2n}$ and $A_{2n-1}B_{2m-1} = B_{2m-1}A_{2n-1}$,

(c) $B_{2n}B_{2m} = B_{2m}B_{2n}$ and $B_{2m-1}B_{2n-1} = B_{2n-1}B_{2m-1}$,

(d) $A_{2n-1}(M) \subset B_{2n}(M)$ and $A_{2n}(M) \subset B_{2n+1}(M)$.

If each $B_q (q = 1, 2, \dots)$ is continuous, then there exists a unique fixed point for two sequences $\{A_p\}$ and $\{B_q\} (p, q = 1, 2, \dots)$.

PROOF. Let x_0 be an arbitrary point in M . By condition (d) there exists a point $x_1 \in M$ such that $A_1x_0 = B_2x_1$. Next we choose a point $x_2 \in M$ such that $A_2x_1 = B_3x_2$. Inductively, we can define by condition (d), the sequence $\{x_n\}$ such that

$$A_{2n-1}x_{2n-2} = B_{2n}x_{2n-1} \text{ and } A_{2n}x_{2n-1} = B_{2n+1}x_{2n}, \quad n \in N. \quad (2.1)$$

First of all we shall show that $\{B_n x_{n-1}\}$ is a Cauchy sequence. By (2.1) and condition (a), we obtain that for all $n \in N$

$$\begin{aligned} d(B_{2n-1}x_{2n-2}, B_{2n}x_{2n-1}) &= d(A_{2n-2}x_{2n-3}, A_{2n-1}x_{2n-2}) \\ &\leq kd(B_{2n-2}x_{2n-3}, B_{2n-1}x_{2n-2}) = kd(A_{2n-3}x_{2n-4}, A_{2n-2}x_{2n-3}) \\ &\leq k^2d(B_{2n-3}x_{2n-4}, B_{2n-2}x_{2n-3}) \leq \dots \leq k^{2n-2}d(B_1x_0, B_2x_1) \end{aligned}$$

and similarly that

$$\begin{aligned} d(B_{2n}x_{2n-1}, B_{2n+1}x_{2n}) &= d(A_{2n-1}x_{2n-2}, A_{2n}x_{2n-1}) \\ &\leq kd(B_{2n-1}x_{2n-2}, B_{2n}x_{2n-1}) \leq \dots \leq k^{2n-1}d(B_1x_0, B_2x_1). \end{aligned}$$

Since $0 < k < 1$, this implies that the sequence $\{B_n x_{n-1}\}$ is a Cauchy sequence. Thus $\{B_n x_{n-1}\}$ converges to some point v in M because M is complete. Now since each $B_q (q \in N)$ is continuous, we obtain that

$$\begin{aligned} B_{2m}v &= B_{2m}(\lim_{n \rightarrow \infty} B_{2n+1}x_{2n}) = \lim_{n \rightarrow \infty} (B_{2m}B_{2n+1}x_{2n}) \\ &= \lim_{n \rightarrow \infty} (B_{2m}A_{2n}x_{2n-1}) = \lim_{n \rightarrow \infty} (A_{2n}B_{2m}x_{2n-1}) \end{aligned}$$

and similarly that $B_{2m+1}v = \lim_{n \rightarrow \infty} (A_{2n+1}B_{2m+1}x_{2n})$ and $B_{2m-1}v = \lim_{n \rightarrow \infty} (A_{2n-1}B_{2m-1}x_{2n-2})$. Hence by condition (c), we have

$$\begin{aligned} d(B_{2m}v, B_{2m+1}v) &= \lim_{n \rightarrow \infty} d(A_{2n}B_{2m}x_{2n-1}, A_{2n+1}B_{2m+1}x_{2n}) \\ &\leq \lim_{n \rightarrow \infty} kd(B_{2n}B_{2m}x_{2n-1}, B_{2n+1}B_{2m+1}x_{2n}) \\ &= kd(B_{2m}v, B_{2m+1}v) \end{aligned}$$

and $d(B_{2m}v, B_{2m-1}v) \leq kd(B_{2m}v, B_{2m-1}v)$ ($m \in N$) in like manner, which implies that $B_mv = B_{m+1}v$ for all $m \geq 1$. Next we shall show that $A_nv = B_nv$ for all $n \leq 1$. By (2.1), conditions (b) and (c), we have

$$\begin{aligned} d(B_{2n+1}B_{2m+2}x_{2m+1}, A_{2n}v) &= d(A_{2m+1}B_{2n+1}x_{2m}, A_{2n}v) \\ &\leq kd(B_{2m+1}B_{2n+1}x_{2m}, B_{2n}v) \\ &= kd(B_{2n+1}B_{2m+1}x_{2m}, B_{2n}v) \end{aligned}$$

Thus letting $m \rightarrow \infty$, we obtain that $d(B_{2n+1}v, A_{2n}v) \leq kd(B_{2n+1}v, B_{2n}v)$ from which it follows that $A_{2n}v = B_{2n+1}v$ for all $n \geq 1$. And since

$$d(A_{2n-1}v, A_{2n}v) \leq kd(B_{2n-1}v, B_{2n}v) \text{ and } d(A_{2n+1}v, A_{2n}v) \leq kd(B_{2n+1}v, B_{2n}v),$$

we obtain that $A_nv = A_{n+1}v = B_{n+1}v = B_nv$ for all $n \in N$. Furthermore, for all $n \in N$, we obtain

$$d(A_{2n}v, A_{2n-1}A_{2n+1}v) \leq kd(B_{2n}v, B_{2n-1}A_{2n+1}v) = kd(A_{2n}v, A_{2n-1}A_{2n+1}v)$$

$$\text{and } d(A_{2n-1}v, A_{2n}A_{2n+1}v) \leq kd(B_{2n-1}v, B_{2n}A_{2n+1}v) = kd(A_{2n-1}v, A_{2n}A_{2n+1}v).$$

Therefore we obtain $u = A_p(u) = B_p(u)$ for all $p \geq 1$ setting $u = A_nv$ because $0 < k < 1$.

Now we shall prove that u is a unique common fixed point of $\{A_p\}$ and $\{B_p\}$. If there exists another point w such that $w = A_p w = B_p w$ for all $p > 1$, then

$$\begin{aligned} d(u, w) &= d(A_{2m-1}u, A_{2m}w) \leq kd(B_{2m-1}u, B_{2m}w) \\ &\leq kd(u, w), \end{aligned}$$

which is a contradiction since $0 < k < 1$. Therefore u is a unique common fixed point of two sequences of self-mappings $\{A_n\}$ and $\{B_n\}$. This completes the proof.

If $S = B_{2n-1}$ and $T = B_{2n}(n = 1, 2, \dots)$, we obtain Theorem 1 as the corollary of Theorem A. Next we obtain the following theorem which is a generalization of Theorem 1 in [4].

THEOREM B. Let (M, d) be a complete metric space and let $\{T_p\}$ ($p = 1, 2, \dots$) be a sequence of mappings from M to M . Suppose that the following conditions are satisfied for all $m \geq n \geq 1$ and $x, y \in M$

(e) there exists a constant h ($h > 1$) such that

$$d(T_{2n-1}x, T_{2n}y) \geq hd(x, y) \text{ and } d(T_{2n}x, T_{2m+1}y) \geq hd(x, y),$$

(f) $T_p T_q = T_q T_p$ (p, q are even or odd respectively).

If every T_n is continuous on M and $T_n(M) = M$ ($n = 1, 2, \dots$), then there exists a unique fixed point for T_n .

PROOF. Set $A_n = I$ (I is the identity map from M to M) in Theorem A. The proof is complete.

REMARK 1. We remark that the mapping $f: X \rightarrow X$ in Theorem 1 of [4] is continuous from the condition of the theorem.

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