A NOTE ON A FUNCTIONAL INEQUALITY

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(Received September 25, 1990 and in revised form March 5, 1991)

ABSTRACT. We prove: If $r_1, \ldots, r_k$ are (fixed) positive real numbers with $\prod_{j=1}^{k} r_j > 1$, then the only entire solutions $\varphi: \mathbb{C} \to \mathbb{C}$ of the functional inequality

$$\prod_{j=1}^{k} |\varphi(r_j z)| \geq \left(\prod_{j=1}^{k} r_j\right) |\varphi(z)|^k$$

are $\varphi(z) = c z^n$, where $c$ is a complex number and $n$ is a positive integer.

KEY WORDS AND PHRASES. Functional inequality, entire functions.

1991 AMS SUBJECT CLASSIFICATION CODE. 39C05.

1. INTRODUCTION.

Inspired by a problem of H. Haruki, who asked for all entire solutions of

$$|\varphi(z + w)|^2 + |\varphi(z - w)|^2 + 2 |\varphi(0)|^2 \geq 2 |\varphi(z)|^2 + 2 |\varphi(w)|^2,$$

(1.1)

J. Walorski [1] proved in 1987 the following interesting proposition:

Let $r > 1$ be a (fixed) real number. Then the only entire solutions $\varphi: \mathbb{C} \to \mathbb{C}$ of the functional inequality

$$|\varphi(rz)| \geq r |\varphi(z)|$$

are

$$\varphi(z) = c z^n,$$  \hfill (1.2)

where $c \in \mathbb{C}$ and $n \in \mathbb{N}$.

As an application of this theorem, Walorsi showed that the only entire functions $\varphi: \mathbb{C} \to \mathbb{C}$ satisfying (1.1) and $\varphi(0) = 0$ are the monomials (1.2). The aim of this note is to prove an extension of Walorski's result by using a method which is (slightly) different from the two approaches presented in [1].

2. MAIN RESULTS.

Theorem. Let $r_1, \ldots, r_k$ be (fixed) positive real numbers with $\prod_{j=1}^{k} r_j > 1$. Then the only entire
solutions \( \varphi : \mathbb{C} \to \mathbb{C} \) of
\[
\prod_{j=1}^{k} |\varphi(r_j z)| \geq \left( \prod_{j=1}^{k} r_j \right) |\varphi(z)|^k
\]  
(2.1)
are the functions \( \varphi(z) + cz^n \), where \( c \) is a complex number and \( n \) is a positive integer.

**PROOF.** Simple calculations reveal that the functions \( \varphi(z) = cz^n \) \((c \in \mathbb{C}, n \in \mathbb{N})\) satisfy (2.1).

Next we assume that \( \varphi \) is an entire solution of inequality (2.1).

Because of \( \prod_{j=1}^{k} r_j > 1 \) we conclude from (2.1) with \( z = 0 \) that \( \varphi \) has at 0 a zero. Let \( n \) be the order of this zero; we define
\[
f(z) = \varphi(z)/z^n,
\]
(2.2)
then \( f \) is an entire function with \( f(0) \neq 0 \). From (2.1) we obtain
\[
\prod_{j=1}^{k} |f(r_j z)| \geq \left( \prod_{j=1}^{k} r_j^{-n} \right) |f(z)|^k.
\]
(2.3)
We suppose that \( f \) has a zero \( z_0 \). By induction it follows from (2.3) that \( z_0/r_1^n \) is a root of \( f \) for all non-negative integers \( m \). From the identity theorem we conclude \( f(z) \equiv 0 \) which contradicts the condition \( f(0) \neq 0 \). Hence \( f \) has no zero which implies that the function
\[
g(z) = \frac{f(z)^k}{\prod_{j=1}^{k} f(r_j z)}
\]
(2.4)
is entire. From (2.3) we conclude
\[
|g(z)| \leq \prod_{j=1}^{k} r_j^n - 1 \quad \text{for all } z \in \mathbb{C},
\]
and Liouville's theorem implies that \( g \) is a constant. Therefore we have
\[
f(z)^k = K \prod_{j=1}^{k} f(r_j z), \quad K \in \mathbb{C}.
\]
(2.5)
Since \( f(0) \neq 0 \) we get from (2.5): \( K = 1 \);

\[
f(z)^k = \prod_{j=1}^{k} f(r_j z).
\]
(2.6)
Differentiation leads to
\[
k f'(z) = \sum_{j=1}^{k} r_j f(r_j z) \frac{f(r_j z)}{f(z)}.
\]
(2.7)
Setting
\[
f'(z) = \sum_{m=0}^{\infty} a_m z^m
\]
(2.8)
we obtain from (2.7) and (2.8):
\[
\sum_{m=0}^{\infty} k a_m z^m = \sum_{m=0}^{\infty} (a_m \sum_{j=1}^{k} r_j^m + 1) z^m,
\]
(2.9)
and comparing the coefficients of $z^n$ yields for all $m \geq 0$:

$$ka_m = a_m \sum_{j=1}^{k} r_j^{m+1}.$$  \hspace{1cm} (2.10)

We assume that there exists an integer $m_0 \geq 0$ such that $a_{m_0} \neq 0$, then we get from the arithmetic mean-geometric mean inequality and from (2.10):

$$\left[ \sum_{j=1}^{k} r_j^{m_0+1} \right]^{1/k} \leq \frac{1}{k} \sum_{j=1}^{k} r_j^{m_0+1} = 1,$$

which contradicts the assumption $\sum_{j=1}^{k} r_j > 1$. Hence, $a_m = 0$ for all $m > 0$. This implies that $f$ is a constant, say $c \in \mathbb{C}$, and therefore we obtain $\varphi(z) = cz^n$.

It is natural to look for all entire functions $\varphi : \mathbb{C} \to \mathbb{C}$ which satisfy the following additive counterpart of inequality (2.1):

$$\left( \sum_{j=1}^{k} \varphi(r_j z) \right) \geq \sum_{j=1}^{k} r_j | \varphi(z) | ,$$  \hspace{1cm} (2.11)

where $r_1, ..., r_k$ are (fixed) positive real numbers with $\sum_{j=1}^{k} r_j > k$. The monomials $\varphi(z) = cz^n (c \in \mathbb{C}, n \in \mathbb{N})$ are solutions of (2.11). Indeed, inequality (2.11) with $\varphi(z) = cz^n$ reduces to

$$\sum_{j=1}^{k} r_j^n \geq \sum_{j=1}^{k} r_j ,$$  \hspace{1cm} (2.12)

which follows immediately from Jensen’s inequality and the assumption $\sum_{j=1}^{k} r_j > k$. By an argumentation similar to the one we have used to establish the theorem it can be shown that the functions $\varphi(z) = cz^n (c \in \mathbb{C}, n \in \mathbb{N})$ are the only entire solutions of (2.11). This provides another extension of Walorski’s result.

If the expression on the left-hand side of (2.11) will be replaced by $\sum_{j=1}^{k} | \varphi(r_j z) |$, then we conclude from the triangle inequality that $\varphi(z) = cz^n (c \in \mathbb{C}, n \in \mathbb{N})$ also solve

$$\sum_{j=1}^{k} | \varphi(r_j z) | \geq \sum_{j=1}^{k} r_j | \varphi(z) | ,$$  \hspace{1cm} (2.13)

where $r_1, ..., r_k$ are (fixed) positive real numbers with $\sum_{j=1}^{k} r_j > k$. We finish by asking: Are there more solutions of (2.13) (if $k > 1$)?

**REFERENCE**

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