ON SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS OF HIGHER ORDER

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ABSTRACT. The classes $T_k(\rho)$, $0 < \rho < 1$, $k > 2$, of analytic functions, using the class $V_k(\rho)$ of functions of bounded boundary rotation, are defined and it is shown that the functions in these classes are close-to-convex of higher order. Covering theorem, arc-length result and some radii problems are solved. We also discuss some properties of the class $V_k(\rho)$ including distortion and coefficient results.

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1. THE CLASS $P_k(\rho)$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in the unit disc $E = \{z: |z| < 1\}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{1 - \rho} \right| d\theta < k\pi, \quad (1.1)$$

where $z = re^{i\theta}$, $k > 2$ and $0 < \rho < 1$. This class has been introduced in [1]. We note that, for $\rho=0$, we obtain the class P_k defined by Pinchuk [2] and for $\rho = 0$, $k = 2$, we have the class P of functions with positive real part. The case $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ .

Also we can write

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\rho) ze^{-it}}{1 - ze^{-it}} d\mu(t), \quad (1.2)$$

where $\mu(t)$ is a function with bounded variation on $[0, 2\pi]$ such that

$$\left. \begin{aligned} \int_0^{2\pi} d\mu(t) &= 2 \\ \int_0^{2\pi} |d\mu(t)| &< k \end{aligned} \right\} \quad (1.3)$$

and

From (1.1), we have the following.

THEOREM 1.1. Let $p \in P_k(\rho)$. Then

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z),$$

where $p_i \in P(\rho)$, $i = 1, 2$.

We now prove.

THEOREM 1.2. The class $P_k(\rho)$ is a convex set.

PROOF. Let $H_1, H_2 \in P_k(\rho)$. We shall show that, for $\alpha, \beta > 0$

$$H(z) = \frac{1}{\alpha + \beta} [\alpha H_1(z) + \beta H_2(z)]$$

belongs to $P_k(\rho)$.

From Theorem 1.1, we can write

$$\begin{aligned} H(z) = \frac{1}{\alpha + \beta} \left[\alpha \left\{ \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \right\} \right. \\ \left. + \beta \left\{ \left(\frac{k}{4} + \frac{1}{2}\right) p_3(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_4(z) \right\} \right], \end{aligned}$$

where $p_i \in P(\rho)$, $i = 1, 2, 3, 4$.

Now, writing $p_i(z) = (1-\rho) h_i(z) + \rho$, $i=1,2,3,4$, see [3], we have

$$\begin{aligned} \frac{H(z)-\rho}{1-\rho} &= \left(\frac{k}{4} + \frac{1}{2}\right) \left\{ \frac{1}{\alpha + \beta} (\alpha h_1(z) + \beta h_3(z)) \right\} - \left(\frac{k}{4} - \frac{1}{2}\right) \left\{ \frac{1}{\alpha + \beta} (\alpha h_2(z) + \beta h_4(z)) \right\} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) f_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) f_2(z), \end{aligned}$$

where f_1 and $f_2 \in P$, since P is a convex set, see [2] and this gives us the required result.

THEOREM 1.3. Let $p \in P_k(\rho)$ and be given by $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then

$$(i) \quad \frac{1}{2\pi} \int_0^{2\pi} |p(re^{i\theta})|^2 d\theta \leq \frac{1 + [k^2(1-\rho)^2 - 1]r^2}{1 - r^2}$$

and

$$(ii) \quad \frac{1}{2\pi} \int_0^{2\pi} |p'(re^{i\theta})| d\theta \leq \frac{k(1-\rho)}{1 - r^2}$$

PROOF. (i) Using Parseval's identity, we have

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |p(re^{i\theta})|^2 d\theta &= \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \\ &= 1 + k^2(1-\rho)^2 \sum_{n=1}^{\infty} r^{2n} = \frac{1 + [k^2(1-\rho)^2 - 1]r^2}{(1 - r^2)}, \end{aligned}$$

where we have used an easily established sharp result $|c_n| \leq k(1-\rho)$, for all $n \geq 1$.

(ii) By using Theorem 1.1, we can write

$$p(z) - \rho = \left(\frac{k}{4} + \frac{1}{2}\right)(1-\rho) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)(1-\rho) h_2(z),$$

where $h_1, h_2 \in P$.

Therefore,

$$p'(z) = \left(\frac{k}{4} + \frac{1}{2}\right)(1-\rho) h_1'(z) - \left(\frac{k}{4} - \frac{1}{2}\right)(1-\rho) h_2'(z) \quad (1.4)$$

Now, for all $h \in P$, we have

$$h'(z) = \frac{2w'(z)}{(1+w(z))^2},$$

where $w(z)$ is a Schwarz function [3], and

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |h'(re^{i\theta})| d\theta &= \frac{1}{2\pi} \int_0^{2\pi} \frac{2|w'(re^{i\theta})|}{|1+w(re^{i\theta})|^2} d\theta \\ &< \frac{2}{1-r^2}. \end{aligned} \quad (1.5)$$

Hence, from (1.4) and (1.5), we have

$$\frac{1}{2\pi} \int_0^{2\pi} |p'(re^{i\theta})| d\theta < \frac{k(1-\rho)}{1-r^2},$$

which is the required result.

From Theorem 1.1 and the properties of the class $P(\rho)$, we immediately have the following.

THEOREM 1.4. Let $p \in P_k(\rho)$. Then

$$\frac{1 - k(1-\rho)r + (1-2\rho)r^2}{1-r^2} < \operatorname{Re} p(z) < \frac{1 + k(1-\rho)r + (1-2\rho)r^2}{1-r^2}$$

THEOREM 1.5. Let $p \in P_k(\rho)$. Then $p \in P$ for $|z| < r_0$, where r_0 is given by

$$r_0 = 2 / [k(1-\rho) + \sqrt{k^2(1-\rho)^2 - 4(1-2\rho)}], \quad \rho \neq \frac{1}{2} \quad (1.6)$$

When $\rho=0$, we obtain the results proved in [2].

2. THE CLASS $V_k(\rho)$

DEFINITION 2.1. Let $V_k(\rho)$ denote the class of analytic and locally univalent functions f in E with normalization $f(0) = 0$, $f'(0) = 1$ and satisfying the condition

$$\frac{(zf'(z))'}{f'(z)} \in P_k(\rho), \quad 0 < \rho < 1, \quad k > 2$$

When $\rho=0$, we obtain the class V_k of functions with bounded boundary rotation. The class $V_k(\rho)$ also generalizes the class $C(\rho)$ of convex functions of order ρ .

It can easily be seen [1] that $f \in V_k(\rho)$ if and only if there

exists $F \in V_k$ such that

$$f'(z) = (F'(z))^{1-\rho} \quad (2.1)$$

In the following, we will study the distortion theorems for the class $V_k(\rho)$. We will use the hypergeometric functions

$$\begin{aligned} G(a, b; c, z) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} (1-zu)^{-b} du, \end{aligned}$$

where $\operatorname{Re} a > 0$ and $\operatorname{Re}(c-a) > 0$. These functions are analytic for $z \in E$ [4]. In addition, we define the functions

$$M_1(a, b; c, r) = \frac{2^{b-1}}{a} [G(a, b; c, -1) - r_1^{-a} G(a, b; c, -r_1^{-1})] \quad (2.2)$$

and

$$M_2(a, b; c, r) = \frac{2^{b-1}}{a} [G(a, b; c, -1) - r_1^a G(a, b; c, -r_1)] ,$$

$$\text{where } r_1 = \frac{1-r}{1+r}$$

THEOREM 2.1. Let $f \in V_k(\rho)$. Then, for $|z| = r$ ($0 < r < 1$), we have

$$M_2(a, b; c, r) < |f(z)| < M_1(a, b; c, r), \quad (2.3)$$

where

$$\left. \begin{aligned} a &= \left(\frac{k}{2} - 1\right)(1-\rho) + 1, \\ b &= 2\rho \\ c &= \left(\frac{k}{2} - 1\right)(1-\rho) + 2 \end{aligned} \right\} , \quad (2.4)$$

and M_1, M_2 are as defined in (2.2).

This result is sharp.

PROOF. Using (2.1) and the well-known bounds for $|F'(z)|$ with $F \in V_k$, see [2], we have

$$\frac{(1-|z|)^{\left(\frac{k}{2}-1\right)(1-\rho)}}{(1+|z|)^{\left(\frac{k}{2}+1\right)(1-\rho)}} < |f'(z)| < \frac{(1+|z|)^{\left(\frac{k}{2}-1\right)(1-\rho)}}{(1-|z|)^{\left(\frac{k}{2}+1\right)(1-\rho)}} \quad (2.5)$$

Let d_r denote the radius of the largest schlicht disk centered at the origin contained in the image of $|z| < r$ under $f(z)$. Then there is a point z_0 , $|z_0| = r$, such that $|f(z_0)| = d_r$. The ray from 0 to $f(z_0)$ lies entirely in the image of E and the inverse image of this ray is a curve in $|z| < r$.

Thus

$$d_r = |f(z_0)| = \int_C |f'(z)| |dz|$$

$$\begin{aligned}
&> \int_C \frac{(1 - |z|)^{\left(\frac{k}{2} - 1\right)(1-\rho)}}{(1 + |z|)^{\left(\frac{k}{2} + 1\right)(1-\rho)}} |dz| \\
&> \int_0^1 \frac{|z| (1-t)^{\left(\frac{k}{2} - 1\right)(1-\rho)}}{(1+t)^{\left(\frac{k}{2} + 1\right)(1-\rho)}} dt \\
&= \int_0^1 |z| \left(\frac{1-t}{1+t}\right)^{\left(\frac{k}{2} - 1\right)(1-\rho)} \frac{dt}{(1+t)^{2(1-\rho)}}
\end{aligned}$$

Let $\frac{1-t}{1+t} = \xi$. Then $\frac{-2}{(1+t)^2} dt = d\xi$.

So

$$\begin{aligned}
|f(z_0)| &> 2^{2\rho-1} \left\{ \int_0^1 \xi^{\left(\frac{k}{2} - 1\right)(1-\rho)} (1+\xi)^{-2\rho} d\xi \right. \\
&\quad \left. - \int_0^1 \frac{1-|z|}{1+|z|} \xi^{\left(\frac{k}{2} - 1\right)(1-\rho)} (1+\xi)^{-2\rho} d\xi \right\}
\end{aligned}$$

Put $\frac{1-|z|}{1+|z|} = \frac{1-r}{1+r} = r_1$ and $\xi = r_1 u$.

This gives

$$\begin{aligned}
|f(z_0)| &> \frac{2^{b-1}}{a} \{G(a, b; c, -1) - r_1^a G(a, b; c, -r_1)\} \\
&= M_2(a, b, c, r),
\end{aligned}$$

where a, b, c and M_2 are respectively defined by (2.4) and (2.2).

Similarly we can calculate the lower bound for $|f(z)|$ and this establishes our result.

Equality is attained in (2.3) for the function $f_0 \in V_k(\rho)$ defined by

$$f'_0(z) = \frac{(1 + \delta_1 z)^{\left(\frac{k}{2} - 1\right)(1-\rho)}}{(1 - \delta_2 z)^{\left(\frac{k}{2} + 1\right)(1-\rho)}}, \quad |\delta_1| = |\delta_2| = 1 \quad (2.6)$$

We now study the behaviour of the integral transform

$$f_\alpha(z) = \int_0^z (f'(\xi))^\alpha d\xi \quad (2.7)$$

for $f \in V_k(\rho)$

This problem has been studied for the class of univalent normalized functions in E and for the close-to-convex functions, see [3]. We have

THEOREM 2.2. Let $f \in V_k(\rho)$, $0 < \rho < 1$, $k > 2$ and let α , $0 < \alpha < 1$ be given. Then $f_\alpha \in V_m$ for $m < [\alpha(1-\rho)(k-2)+2]$.

PROOF. From (2.1), we have

$$f'(z) = (F'(z))^{1-\rho}, \quad F \in V_k$$

Now

$$\begin{aligned} f'_\alpha(z) &= (f'(z))^\alpha = (F'(z))^{\alpha(1-\rho)} \\ &= \exp \int_{-\pi}^{\pi} -\log(1-\xi e^{-it}) \alpha(1-\rho) \, d\mu(t) \\ &= \exp \int_{-\pi}^{\pi} -\log(1-\xi e^{-it}) \, d\mu(t), \end{aligned}$$

where $d\mu(t) = \alpha(1-\rho) \, d\mu(t) + [1 - \alpha(1-\rho)] \frac{dt}{\pi}$

Also

$$\int_{-\pi}^{\pi} d\mu(t) = \alpha(1-\rho) \int_{-\pi}^{\pi} d\mu(t) + \frac{1-\alpha(1-\rho)}{\pi} \int_{-\pi}^{\pi} dt = 2,$$

and

$$\begin{aligned} \int_{-\pi}^{\pi} |d\mu(t)| &< \alpha(1-\rho) \int_{-\pi}^{\pi} |d\mu(t)| + \frac{1-\alpha(1-\rho)}{\pi} \int_{-\pi}^{\pi} dt \\ &< \alpha(1-\rho)k + 2[1 - \alpha(1-\rho)]. \end{aligned}$$

Hence the result.

We note that f_α is univalent for $\alpha < \frac{2}{(1-\rho)(k-2)}$, since V_m consists of univalent functions for $2 < m < 4$. Hence f_α is univalent even if f is not univalent provided $\alpha < \frac{2}{(1-\rho)(k-2)}$.

Using the standard technique, we can easily prove the following.

THEOREM 2.3. Let $g, h \in V_k(\rho)$ and let $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1 - \rho$. Then

$$H(z) = \int_0^z (g'(t))^\alpha (h'(t))^\beta \, dt$$

is convex of order $\rho_1 = (1 - \frac{\alpha+\beta}{1-\rho})$ for $|z| < r_1$,

where

$$r_1 = \frac{1}{2} [k - \sqrt{k^2 - 4}] \quad (2.8)$$

The result is sharp when

$$g'(z) = h'(z) = \left[\frac{(\frac{k}{2} - 1)(1-\rho)}{(1-z)^{\frac{k}{2}} (\frac{k}{2} + 1)(1-\rho)} \right].$$

We now prove the following.

THEOREM 2.4. Let $f: f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V_k(\rho)$. Then, for all $n > 3$, $2 < k < \infty$.

$$|a_n| < [k^2(1-\rho)^2 + k(1-\rho)] 2^{-2\rho} \left(\frac{2n}{3}\right)^{(1-\rho)(\frac{k}{2} + 1) - 2}$$

The function f_0 defined by (2.6) shows that the exponent

$[(1-\rho)(\frac{k}{2} + 1) - 2]$ is best possible.

PROOF. By definition, we have

$$(zf'(z))' = f'(z) p(z), \quad p \in P_k(\rho).$$

Set

$$\begin{aligned} F(z) &= (z(zf'(z)))' \\ &= f'(z) [p^2(z) + zp'(z)]. \end{aligned}$$

For $z = re^{i\theta}$, we have

$$n^3 |a_n| < \frac{1}{2\pi r^{n-3}} \int_0^{2\pi} |f'(z)| |p^2(z) + zp'(z)| d\theta$$

Using (2.5) and theorem 1.3, we obtain

$$\begin{aligned} n^3 |a_n| &< \frac{1}{r^{n-3}} \frac{(1-\rho)(\frac{k-2}{2})}{(1-\rho)(\frac{k+2}{2})} \frac{\{1 + [k^2(1-\rho)^2 - 1]r^2 + k(1-\rho)\}}{1 - r^2} \\ &= \frac{1}{r^{n-3}} \frac{(1-\rho)(\frac{k-2}{2})-1}{(1-\rho)(\frac{k+2}{2})+1} \frac{\{1 + k(1-\rho) + [k^2(1-\rho)^2 - 1]r^2\}}{(1-r)} \end{aligned}$$

Let $r = 1 - \frac{3}{n}$, $n > 3$. Then

$$\begin{aligned} n^3 |a_n| &< [k^2(1-\rho)^2 + k(1-\rho)] e^3 \cdot (2 - \frac{3}{n})^{(1-\rho)(\frac{k-2}{2})} (\frac{n}{3})^{(1-\rho)(\frac{k+2}{2})+1} \\ &= [k^2(1-\rho)^2 + k(1-\rho)] e^3 \cdot (\frac{n}{3})^{[(1-\rho)(\frac{k+2}{2})-2]} \cdot \frac{n^3}{27} (2 - \frac{3}{n})^{(1-\rho)(\frac{k}{2} - 1)-1} \end{aligned}$$

Thus, for $n > 3$,

$$|a_n| < [k^2(1-\rho)^2 + k(1-\rho)] (2)^{-2\rho} \cdot (\frac{2n}{3})^{(1-\rho)(\frac{k}{2} + 1) - 2}$$

THEOREM 2.5. Let $f \in V_k(\rho)$, $\rho \neq 1/2$. Then f maps $|z| < r_0$ onto a convex domain where r_0 is given by (1.6). The function f_0 , defined by (2.6) shows that this result is sharp.

The proof is straightforward and follows immediately from the definition and Theorem 1.5.

Furthermore it can easily be shown that if $f \in V_k(\rho)$, then f is convex of order ρ for $|z| < r_1$ where r_1 is given by (2.8).

3. THE CLASS $T_k(\rho)$.

A class T_k of analytic functions related with the class V_k has been introduced and studied in [5]. We now define the following.

DEFINITION 3.1. Let f with $f(0) = 0$, $f'(0) = 1$ be analytic in E . Then $f \in T_k(\rho)$, $k > 2$, $0 < \rho < 1$, if there exists a function $g \in V_k(\rho)$ such that $\frac{f'(z)}{g'(z)} \in P$ for $z \in E$.

Note that $T_k(0) = T_k$ and $T_2(0)$ is the class of close-to-convex functions.

THEOREM 3.1. Let $f \in T_k(\rho)$. Then

$$\{f(z)\} > M_2(a+1, b; c+1, r),$$

where $M_2(a, b; c, r)$ is defined by (2.2) and a, b, c are given by (2.4). This result is sharp.

PROOF. Since $f \in T_k(\rho)$, we can write

$$f'(z) = g'(z) h(z), \quad g \in V_k(\rho), \quad h \in P.$$

It is well-known that for $h \in P$

$$|h(z)| > \frac{1 - |z|}{1 + |z|} \quad (3.1)$$

Thus, using (3.1) and (2.5), we have

$$|f'(z)| > \frac{(1 - |z|)^{\left(\frac{k}{2} - 1\right)(1-\rho) + 1}}{(1 + |z|)^{\left(\frac{k}{2} + 1\right)(1-\rho) + 1}}$$

Proceeding in the same way as in Theorem 2.1, we obtain the required result.

REMARK 3.1. When $\rho=0$, $f \in T_k$ and since in this case $b = 0 < 1$, $c = 1+a-b$, we have $G(a, b; c, -1) = 1$. Letting $r \rightarrow 1$, with $\rho = 0$, in Theorem 3.1, we see that the image of E under functions f in T_k contains the schlicht disk $|z| < \frac{1}{k+2}$.

We now give a necessary condition for a function f to belong to the class $T_k(\rho)$.

THEOREM 3.2. Let $f \in T_k(\rho)$. Then, with $z = re^{i\theta}$ and $\theta_1 < \theta_2$; $0 < \rho < 1$,

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -k(1-\rho) \frac{\pi}{2}.$$

PROOF. We can write

$$f'(z) = (g_1'(z))^{1-\rho} (h_1(z))^{1-\rho}, \quad \text{for some } g_1 \in V_k, h_1 \in P.$$

So

$$f'(z) = (g_1'(z) h_1(z))^{1-\rho} = (f_1'(z))^{1-\rho}, \quad (3.3)$$

for some $f_1 \in T_k$.

Hence

$$\frac{(zf'(z))'}{f'(z)} = (1-\rho) \frac{(zf_1'(z))'}{f_1'(z)} + \rho$$

The required result follows on noting that, for $\theta_1 < \theta_2$, $f_1 \in T_k$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf_1'(z))'}{f_1'(z)} d\theta > -\frac{k}{2} \pi, \quad \text{see [5].}$$

REMARK 3.2. In [1], Goodman introduced the class $K(\beta)$ of normalized analytic functions which are close-to-convex of order $\beta > 0$ and showed that if f is analytic in E and $f'(z) \neq 0$, then for $\beta > 0$, $f \in K(\beta)$ if for $z = re^{i\theta}$ and $\theta_1 < \theta_2$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re} \frac{(zf'(z))'}{f'(z)} d\theta > -\beta\pi$$

When $0 < \beta < 1$, $K(\beta)$ consists of univalent functions, whilst if $\beta > 1$, f need not even be finitely-valent.

We note that Theorem 3.2 shows that

$$T_k(\rho) \subset K\left(\frac{k(1-\rho)}{2}\right).$$

Hence $T_k(\rho)$ consists entirely of univalent functions if $2 < k < \frac{2}{1-\rho}$. It also follows easily from the definition that the class $T_k(\rho)$ forms a subset of a linear-invariant family of order $[\frac{k}{2}(1-\rho)+1]$.

Using the method of Clunie and Pommerenke as modified by Thomas [7], we can easily prove the following:

THEOREM 3.3. Denote by $L(r, f)$ the length of the image of the circle $|u|=r$ under f and by $M(r) = \max_{\theta} |f(re^{i\theta})|$. Then, for $0 < r < 1$,

$$L(r) < A(k, \rho) M(r) \log \frac{1}{1-r},$$

where $A(k, \rho)$ is a constant depending only on k and ρ .

Let $P_{\alpha, 1}$ denote the class of functions $p(z)$ in E given by

$$p(z) = 1 + c_1 z + c_2 z^2 + \dots$$

which satisfy the inequality

$$|p(z) - \frac{1}{2\alpha}| < \frac{1}{2\alpha}, \quad 0 < \alpha < 1$$

The class $P_{\alpha,1}$ has been introduced in [8] and it is shown there that, for $p \in P_{\alpha,1}$, $|z| = r < 1$.

$$\left| \frac{p'(z)}{p(z)} \right| < \frac{(1+c)}{(1+cr)(1-r)}, \quad (3.4)$$

where $c = 1 - 2\alpha$

We now prove the following.

THEOREM 3.4. Let $g \in V_k(\rho)$ and let $\frac{f'(z)}{g'(z)} \in P_{\alpha,1}$. Then f is a convex function of order ρ for $|z| < r$ where $r \in (0,1)$ is the least positive root of the equation

$$(1-\rho)cx^3 - [(\rho+c) + ck(1-\rho)]x^2 + [\rho(k-c) - (1+k)]x + (1-\rho) = 0$$

PROOF. We can write

$$f'(z) = (g_1'(z))^{1-\rho} p(z), \quad g_1 \in V_k, \quad p \in P_{\alpha,1}$$

So

$$\operatorname{Re} \left[\frac{(zf'(z))'}{f'(z)} - \rho \right] > (1-\rho) \operatorname{Re} \left[\frac{(zg_1'(z))'}{g_1'(z)} \right] - \left| \frac{zp'(z)}{p(z)} \right|$$

Using Theorem 1.4 with $\rho = 0$ and (3.4), we have the required result.

Furthermore, if

$$T(r) = (1-\rho)cr^3 - [(\rho+c) + ck(1-\rho)]r^2 + [\rho(k-c) - (1+k)]r + (1-\rho),$$

then we note that

$$T(0) = (1-\rho) > 0$$

$$T(1) = -2\rho c - 2\rho - ck(1-\rho) - k(1-\rho) < 0$$

Thus $r \in (0,1)$.

COROLLARY 3.1. When $\alpha = 0$, $c = 1$ and $\rho = 0$, $f \in T_k$. Thus f maps $|z| < r = \frac{1}{2}[(k+2) - \sqrt{k^2 + 4k}]$ onto a convex domain and this result is sharp, see [5].

COROLLARY 3.2. When $\rho=0$, $\alpha = \frac{1}{2}$, and then we have $|\frac{f'(z)}{g'(z)} - 1| < 1$ for $g \in V_k$. Then f is convex for $|z| < r = \frac{1}{k+1}$. For $k=4$, V_k consists of univalent functions and in this case $r = \frac{1}{5}$. This result is proved in [8]. For $\alpha = 0$, $k = 4$ and $\rho = 0$, we obtain the known result $r = 3 - 2\sqrt{2}$ of Ratti [9] and when $k = 2$, we have the well-known result giving us the radius of convexity for close-to-convex functions.

Finally we have

THEOREM 3.5. Let $f \in V_k(\rho)$ and let

$$F(z) = \frac{1}{1+m} z^{1+m} |z^m f(z)|', \quad m = 1, 2, 3, \dots$$

Then $F \in T_k(\rho)$ for all $|z| < r_2$, where, for $(1-2\rho-m) \neq 0$, $0 < \rho < 1$,

$$r_2 = 2(1+m) / [(1-\rho)k + \sqrt{(1-\rho)^2 k^2 - 4(1-2\rho-m)(1+m)}],$$

The proof is straightforward when we note that

$$\operatorname{Re} \frac{F'(z)}{f'(z)} = \frac{1}{1+m} \left[\operatorname{Re} \frac{(zf'(z))'}{f'(z)} + m \right]$$

and then use theorem 1.4.

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