A NOTE ON THE SUPPORT OF RIGHT INVARIANT MEASURES

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ABSTRACT. A regular measure μ on a locally compact topological semigroup is called right invariant if $\mu(Kx) = \mu(K)$ for every compact K and x in its support. It is shown that this condition implies a property reminiscent of the right cancellation law. This is used to generalize a theorem of A. Mukherjea and the author (with a new proof) to the effect that the support of an r^* -invariant measure is a left group iff the measure is right invariant on its support.

KEY WORDS AND PHRASES. Topological semigroup, left group, right invariant (Borel) measure, r^* -invariant measure, support of a Borel measure, locally compact semigroup.

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1. INTRODUCTION.

In what follows S will denote a T_2 locally compact topological semigroup (jointly continuous multiplication) and μ a positive regular (Radon) measure on the Borel σ -algebra of S with support $F = \{s \in S; \text{ for every open } V \supset s, \mu(V) > 0\}$, as in [1] and [2]. We shall use the notation $Bx^{-1} = t_x^{-1}(B) = \{s \in S; sx \in B\}$, t_x denoting the right continuous translation $s \rightarrow sx$. The measure is called r^* -invariant on S if $\mu(Bx^{-1}) = \mu(B)$ for all Borel B and x in S . Such measures received considerable attention in the past in connection with the (still unsolved) conjecture of L.N. Argabright (Proc. Amer. Math. Soc. 17 (1966), 377-382) that their support is a left group i.e., F is left simple ($Fx = F$ for all x in F) and right cancellative (equivalently, if it is left simple and contains an idempotent element). The measure μ is called right invariant on its support if

$$\mu(Kx) = \mu(K) \text{ for every compact } K \subset F \text{ and every } x \in F \quad (1.1)$$

In [1] A. Mukherjea and the author proved the "rather tight" result

THEOREM 1. The support of an r^* -invariant measure on S is a left group iff the measure is right invariant on its support.

Professor Mukherjea in a meeting at University of South Florida asked the questions (i) whether the "intriguing" condition (1.1) (introduced by himself) implies some sort of right cancellation on F in view of the fact proven by Rigelhof [3] that (1.1) plus that the t_x 's, $x \in F$, are open maps, imply right cancellation on F . (ii) Whether Theorem 1 (above) can be substantially generalized. In this note we show: As for question (i) indeed there is a "generalized" right cancellation on S (See

Lemma 1, below) but as for question (ii), Theorem 1 cannot substantially be generalized except that we may only assume that $\mu(Bx^{-1}) \geq \mu(B)$ for every Borel B and every $x \in F$. (Unlike condition (1.1), no extra generality is obtained whether we assume $B \subset S$ or $B \subset F$). Moreover, our proof (although patented on that of [1]) does not use the functional analytic apparatus of [1] since it uses a version of cancellation from the intrinsic properties of the measure.

2.

We begin by showing in what sense S is pre-right cancellable mod F .

LEMMA 1. Let μ be right invariant on its support (i.e., μ satisfies (1.1)). Then

- (i) If for $f_1, f_2, f_3 \in F$, $f_1 f_2 = f_3 f_2$, then $ff_1 = ff_3$ for every $f \in F$ that is, we can cancel on the right by premultiplying by any element of the support.
- (ii) If F is also a right ideal of S , then for $s_1, s_3 \in S$, $f_2 \in F$, the equation $s_1 f_2 = s_3 f_2$ implies $fs_1 = fs_3$ for all $f \in \overline{FF} = \text{closure}(FF)$ and in particular for any idempotent element $e \in F$.

PROOF. We shall argue by contradiction as in Rigelhof [3, p. 175]. We prove (ii): (The proof of (i) is done similarly). Assume $s_1 f_2 = s_3 f_2$ but $fs_1 \neq fs_3$ so that we can find disjoint compact neighborhoods U and V respectively of these two distinct points (with f some point in $\overline{FF}$). Now $Us_1^{-1} \cap Vs_3^{-1}$ must contain a compact neighborhood W of f which in turn must contain a right translate of some compact neighborhood of the form $K\phi$ for some $\phi \in F$ ($f \in \overline{FF}$), i.e.,

$$K\phi \subset W \subset Us_1^{-1} \cap Vs_3^{-1}, \text{ so that}$$

$$\mu(K) + \mu(K) = \mu(K\phi s_1) + \mu(K\phi s_3) = \mu(K\phi s_1 \cup K\phi s_3 f_2) = \mu(K \cup K)\phi s_1 f_2 = \mu(K),$$

which is a contradiction.

COROLLARY 1. Let μ satisfy (1.1). Then

- (i) For any pair of idempotents $e_1, e_2 \in F$, we have $e_1 e_2 = e_1$ so that the idempotents in F form a left-zero subsemigroup of F .
- (ii) For any idempotent $e \in F$, eF is right cancellable.
- (iii) If $xyzyx = zyx$ for $x, y, z \in F$, then zy is idempotent.

PROOF. (i): It follows since $e_1 e_2 = e_1 e_2 e_2$ and by Lemma 1 we may cancel e_2 by premultiplying by e_1 and use the fact that e_1 is idempotent. (ii): Similarly by the above Lemma. (iii): First cancel x by premultiplying by y and then cancel zy by premultiplying by z and obtain $zyzy = zy$.

Now we are ready to give the generalization of Theorem 1 as follows:

THEOREM 2. Suppose μ satisfies

$$\mu(Bf^{-1}) \geq \mu(B) \text{ for every Borel } B \text{ and every } f \in F \quad (2.1)$$

Then F is a left group iff μ satisfies (1.1).

PROOF. Clearly (1.1) plus inner regularity of μ imply $\mu(Bf^{-1}) \leq \mu(B)$ for all Borel B and $f \in F$ so that we have $\mu(Bf^{-1}) = \mu(B)$ for every $f \in F$ and Borel B . Also (2.1) implies that $\overline{Ff} = F$ for all $f \in F$. In the proof of Theorem 1 in [1],

we produced an idempotent e in Fa , for $a \in F$, so that $Fe = \overline{Fe} = F \subset Fa$ and so $Fa = F$ for all $a \in F$ (cf. [1], p. 974). Now, the same proof goes through without any difficulty except that instead of the right cancellation on Fa , $a \in F$, we use Corollary 1 (iii) above.

We give next a result summarizing certain important conditions on F and μ that are equivalent to F being a left group.

COROLLARY 2. For a locally compact second countable semigroup S admitting an r^* -invariant measure μ , these are equivalent:

- (i) F is right cancellable
- (ii) μ is right invariant on its support, i.e., satisfies (1.1)
- (iii) S is pre- right cancellative with respect to F , i.e., $s_1 s_2 = s_3 s_2$ with $s_1, s_2, s_3 \in F$, implies $fs_1 = fs_3$ for all $f \in F$.
- (iv) F is a left group.
- (v) F has the right translations t_f closed for all $f \in F$.
- (vi) F has the right translations open and μ satisfies (1.1).

REMARK. It is not known to our knowledge if (v) and (iv) are equivalent in the absence of second countability.

PROOF. Most of these follow from Theorem 1 or Theorem 2. Note that right cancellation implies that t_f are one-to-one and for compact K , $Kxx^{-1} \cap F = Kx$, so that right invariance on its support follows from r^* -invariance, so (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) (cf. Theorem 2). Since F is metrizable being regular the technique in [4] for producing an idempotent in Fx applies and thus F becomes a left group. By the result of Rigelhof, (vi) implies (i). ([3] p. 175). For (iii), see our Lemma 2, below.

REMARK. The following will show the "tightness" of the conditions of Theorem 2. It is well known that a property that "melds" naturally (at first sight) with condition (1.1) is that of lower r^* -invariance, i.e., that $\mu(Bx^{-1}) \leq \mu(B)$ for all Borel $B \subset S$ and $x \in S$, for it and (1.1) are equivalent to the condition (cf. [2] and [5, p. 92])

$$\mu(Kx) \geq \mu(K) \quad \text{for all compact } K \subset S \text{ and } x \in S \text{ with} \quad (2.2)$$

this inequality becoming equality whenever K and x are in F .

This condition (2.2) implies that F is a right ideal and $Fe = F$ for every idempotent $e \in S$, but these are not enough to make Theorem 2 valid, for the example of $[0, \infty)$ with addition and Lebesgue measure shows that μ is not r^* -invariant (it does not satisfy (2.1) of Theorem 2). However this S is pre- right cancellative as the following Lemma generally indicates.

LEMMA 2. Suppose μ satisfies (2.2) and suppose that $s_1 s_2 = s_3 s_2$ for $s_1, s_2, s_3 \in S$. Then $fs_1 = fs_3$ for all $f \in \overline{FF}$. If moreover $s_1 s_2 \in F$, then $fs_1 = fs_3$ for all $f \in F$.

PROOF. Suppose first that $s_1 s_2 \notin F$. Then the second equality in the proof of Lemma 1 (ii) with f_2 replaced by s_2 becomes less or equal and the last remains equality and thus a contradiction obtains. Next assume that $s_1 s_2 = s_3 s_2$ and $s_1 s_2 \in F$. Again, as before (See proof of Lemma 1) there are disjoint compact neighborhoods

U, V of fs_1, fs_3 respectively, such that the intersection of Us_1^{-1} and Vs_3^{-1} contain a compact neighborhood W of f (we use $W \cap F$ instead of W). Then we have again the inequality

$$\mu(W) + \mu(W) \leq \mu(Ws_1) + \mu(Ws_3) \leq \mu(Ws_1 \cup Ws_3)s_2 = \mu(W \cup W)s_1s_2 = \mu(W),$$

again a contradiction.

REMARK. The most difficult part in problems involving the nature of the support F is producing an idempotent element in Fx or in F itself. For this, it would be interesting to have a "survey paper" giving all known methods for producing an idempotent in the presence of measure and/or topological invariance conditions. Apart for having some compact subsemigroup or a compact fiber $xx^{-1} \neq \emptyset$ or a two-sided version of (2.2) and a subsemigroup of positive finite inner measure, we know only the technique in [1] which is in some sense an adoption of a method of Gelbaum and Kalisch (Canad. J. of Math. 4 (1952), 396-403), and the technique of [4] which needs metrizability!. (For example, when the t_x are closed mappings, can the "onto-ness" of the t_x be used to prove that the operator $\pi_s f(x) = f(xs)$ on $L_2(S, \mu)$ is onto in the non-second countable case? (that will suffice to prove that the support of an r^* -invariant measure is a left group when the t_x 's are closed).

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