ORTHOGONAL BASES IN A TOPOLOGICAL ALGEBRA ARE SCHAUDER BASES

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ABSTRACT. In a topological algebra with separately continuous multiplication, the result quoted in the title is proved.

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1. INTRODUCTION.
A topological algebra $A$ is a linear associative algebra over complex scalars which is a Hausdorff topological vector space (TVS) in which multiplication is separately continuous, i.e., for each $x \in A$, the operators $L_x$ and $R_x$, $L_x y = xy$, $R_x y = yx$ $(y \in A)$, are continuous. A basis $(e_n)$ in $A$ is Schauder (respectively b-Schauder) if the functionals $e_n$, $e_n(x) = \alpha_n$ (where $\sum \alpha_n e_n$), are continuous (respectively bounded i.e. map bounded sets to bounded sets). An orthogonal basis is a basis $(e_n)$ satisfying $e_n e_m = \delta_{nm} e_n$ for all $n, m$.

Recently S. El-Helaly and T. Husain [1] showed that an orthogonal basis in $A$ is Schauder if multiplication is jointly continuous (i.e. continuous as a bilinear map on $A \times A$). Now joint continuity is a very stringent requirement. In fact, abundance of examples have forced upon some other weaker modes of continuity in literature. Multiplication in $A$ is hypocontinuous (respectively sequentially jointly continuous) if given a $\alpha$-neighborhood $U$ and a bounded set $B$, there is a $\alpha$-neighborhood $V$ such that $BV \subseteq U$, $VB \subseteq U$ (respectively for sequences $(x_n)$, $(y_n)$ in $A$, $x_n \to x$, $y_n \to y$ imply $x_n y_n \to xy$). In a topological algebra, joint continuity gives hypocontinuity which in turn implies sequential joint continuity; and if $A$ is barrelled (respectively complete matrixizable or m-convex), multiplication is hypocontinuous (respectively jointly continuous). We extend the above result of El-Helaly and Husain in its final form by modifying their arguments, and also obtain its variant in a more general frame-work.
2. MAIN RESULTS.

THEOREM. Let $A$ be a Hausdorff TVS that is an algebra

(1) If $A$ is a topological algebra, then every orthogonal basis in $A$ is Schauder.

(2) If multiplication in $A$ is sequentially separately continuous (i.e. for a sequence $(x_n)$ in $A$, $x_n \to 0$ implies $x_n y \to 0$, $y x_n \to 0$ for all $y$), then every orthogonal basis in $A$ is $b$-Schauder.

PROOF. Let $(e_n)$ be an orthogonal basis in $A$. Let $n \in \mathbb{N}$ be fixed. Orthogonality applied to the expansion $x = \sum_{n=1}^{\infty} e_n(x)e_n$ implies that $e_n x = e_n(x)e_n = xe_n$ for all $x$ in $A$. Choose a balanced $\epsilon$-neighborhood $U$ such that $e_n U$. Let $r = \inf \{d > 0: e_n \epsilon U\}$. Then $r > 1$.

(1) Let $(x_n)$ be a net in $A$ such that $\lim x_n = 0$. Hence $\lim e_n x_n = 0$. Given an $\epsilon > 0$, there is an $\alpha_0$ such that $e_n^*(x_\alpha)e_n = x_\alpha e_n \epsilon (\epsilon U)$ for all $\alpha \geq \alpha_0$. As $U$ is balanced, $|e_n^*(x_\alpha)| e_n \epsilon (\epsilon U)$ for $\alpha \geq \alpha_0$. Hence by the definition of $r$, $|e_n^*(x_\alpha)| r \geq r > 1$, and so $|e_n^*(x_\alpha)| < \epsilon$ for all $\alpha \geq \alpha_0$. Thus $\lim e_n^*(x_\alpha) = 0$.

(2) Since a subset in a TVS is bounded iff each of its countable subset is bounded, it is sufficient to show that $e_n^*$ maps a bounded sequence $(x_k)$ to a bounded sequence. Now for any sequence $r_k \to 0$, $x_k/r_k \to 0$. By sequential separate continuity of multiplication, $e_n(x_k/r_k \to 0)$. Hence $(e_n^*(x_k) e_n) = 1$ is bounded, and for all $k$, $e_n^*(x_k) e_n \epsilon U$ for some $\lambda = \lambda(n, U) > 0$. Again by definition of $r$, $|e_n^*(x_k)| \leq \frac{\epsilon}{\lambda}$ for all $k$.

REMARKS. (1) It follows that Corollaries 1.2 and 2.2 in [1] hold for any topological algebra.

(2) In a topological algebra, a basis which is not orthogonal need not be Schauder even if multiplication is sequentially jointly continuous. The algebra $1^1$ of summable scalar sequences with weak topology $\sigma = \sigma(1^1, e_0)$ is a topological algebra in which multiplication (pointwise) is sequentially jointly continuous. Let $e_n = (e_{nm})_{m=1}^{\infty}$. Then $(f_n)$ defined by $f_1 = e_1$, $f_n = (-1)^{n+1} e_1 + e_n(n \geq 2)$ is a basis which is not Schauder [2]. In fact, $f_1^* = e_1^* + e_2^* - e_3^* + e_4^* - e_5^* + ...$, $f_n^* = e_n^*(n \geq 2)$.

REFERENCES


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