ON STRICTLY CONVEX AND STRICTLY 2-CONVEX 2-NORMED SPACES II

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ABSTRACT. In this paper a new duality mapping is defined, and it is our object to show that there is a similarity among these three types of characterizations of a strictly convex 2-normed space. This enables us to obtain more new results along each of two types of characterizations. We shall also investigate a strictly 2-convex 2-normed space in terms of the above two different types.

KEY WORDS AND PHRASES: Linear 2-normed space, strict convexity, strict 2-convexity, 2-semi-inner product, bounded linear 2-functional, duality mapping.

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1. INTRODUCTION.

This article is a continuation of the paper by Lin [11] where we investigated characterizations of strictly convex and strictly 2-convex 2-normed spaces which were initiated by Diminnie, Gähler and White [5,6]. The concept of strictly convex 2-normed space is 2-dimensional analogue of that of strictly convex normed linear space, an important space in functional analysis, and a strictly 2-convex 2-normed space is its natural generalization. A strictly convex 2-normed space is strictly 2-convex (Theorem 8 [6] and Theorem 3 [11]). But the converse is not generally true (Example 2 [6]). Note, however, that strict 2-convexity together with a certain condition is equivalent to strict convexity (Theorem 3 [11]). Most elementary 2-normed spaces originated by Gähler [7] are strictly convex. For example, a 2-normed space of dimension 2, and a 2-inner product space [6]. A strictly convex normed linear space may be characterized in terms of norms by Giles [8], semi-inner products by Berkson [1], or duality mappings by Browder [2], Gudder and Strawther [9] and many others. In this paper a new duality mapping is defined, and it is our object to show that there is a similarity among these three types of characterizations of a strictly convex 2-normed space. This enables us to obtain more new results along each of two types of characterizations. We shall also investigate a strictly 2-convex 2-normed space in terms of the above two different types.

Let X denote a real linear space of dimension greater than one, the following standard definition was introduced in [7]. If $\|\cdot, \cdot\|$ is a real function on $X \times X$, then X is called a 2-normed space with a 2-norm $\|\cdot, \cdot\|$ if the following conditions are satisfied:

- (i) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
- (ii) $\|x, y\| = \|y, x\|$;
- (iii) $\|ax, y\| = |a| \|x, y\|$ for any real a ; and

$$(iv) \quad \|x + y, z\| \leq \|x, z\| + \|y, z\|.$$

Let X be a 2-normed space throughout this paper. If $x, y, z \in X$ are nonzero vectors, we denote by $V(x)$, $V(x, y)$ and $V(x, y, z)$ the linear manifolds of X generated by x , x and y , x , y and z , respectively.

2. STRICTLY CONVEX 2-NORMED SPACES.

Recall from [5] that X is said to be strictly convex if $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| = 1$ for $z \notin V(x, y)$

implies $x = y$. In this section we shall give several characterizations of this space in terms of 2-semi-inner products and duality mappings. But first we need the following lemma which is essential to our consequent theorems, and which is a portion of Theorem 1 in [11] plus three new statements (8), (9), and (10).

LEMMA 1. The following ten statements are equivalent:

- (1) X is strictly convex;
- (2) $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ implies $x = y$;
- (3) $\|x + y, z\| = \|x, z\| + \|y, z\|$ for $z \notin V(x, y)$ implies $x = by$ for some $b > 0$;
- (4) $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| \neq 0$ for $x \neq y$ implies $z = d(x - y)$ for some $d \neq 0$;
- (5) $\|x + ay, z\| = 2\|x, z\|$ for $z \notin V(x, y)$ and $a = \|x, z\|/\|y, z\|$ implies $x = ay$;
- (6) $\|x + y, z\| = \|x, z\| + \|y, z\|$ for $z \notin V(x, y)$ implies $\|y, z\|x = \|x, z\|y$;
- (7) $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| \neq 0$ for $x \neq y$ implies $\|x, y\| \neq 0$ and $z = \pm\|x, z\|(x - y)/\|x, y\|$;
- (8) $\|w + x, z\| = \|w + y, z\| \neq 0$ for all $w \in X$ implies $x = y$;
- (9) $\|x - y, z\| = |\|x, z\| - \|y, z\||$ for $z \notin V(x, y)$ implies $x = sy$ for some $s > 0$;
- (10) $\|x - y, z\| = |\|x, z\| - \|y, z\||$ for $z \notin V(x, y)$ implies $\|y, z\|x = (\|x - y, z\| + \|y, z\|)y$.

PROOF. The equivalence of (1) through (7) was proved in (Theorem 1 [11]), and that (10) $\Rightarrow$ (9) is obvious. That (9) $\Rightarrow$ (3) is clear after we verify the implication (6) $\Rightarrow$ (10).

(6) $\Rightarrow$ (10): We may write the relation in (10) as $\|x, z\| = \|x - y, z\| + \|y, z\|$. So $\|y, z\|(x - y) = \|x - y, z\|y$ by (6) and the result follows.

(2) $\Rightarrow$ (8): Let $w = x$ and $w = y$ in (8), then $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ implies $x = y$ by (2).

(8) $\Rightarrow$ (2): Suppose that $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ and $x \neq y$, then $\|w + x, z\| = \|w + y, z\| \neq 0$ for some $w \in X$ (indeed, $w = x$ and $w = y$) and $x \neq y$, i.e., (8) does not hold.

The concept of 2-semi-inner product defined by Siddiqui and Rizvi [14] is 2-dimensional analogue of that of the usual semi-inner product in functional analysis. A 2-semi-inner product is a mapping $[\cdot, \cdot | \cdot]$ on $X \times X \times X$ into real numbers such that

- (i) $[x + x', y | z] = [x, y | z] + [x', y | z]$;
- (ii) $[ax, y | z] = a[x, y | z]$ for any real a ;
- (iii) $[x, x | z] \geq 0$; $[x, x | z] = 0$ if and only if x and z are linearly dependent; and
- (iv) $[x, y | z]^2 \leq [x, x | z][y, y | z]$.

Every 2-normed space can be made into a 2-semi-inner product space, and the norm is given by $\|x, y\| = [x, x | y]^2$ [14].

THEOREM 1. The following nine statements are equivalent:

- (1) X is strictly convex (in the sense of Lemma 1);
- (2) $[x, y | z] = \|x, z\| = \|y, z\| = 1$ for $z \notin V(x, y)$ implies $x = y$;
- (3) $[x, y | z] = \|x, z\|^2 = \|y, z\|^2$ for $z \notin V(x, y)$ implies $x = y$;
- (4) $[w, x | z] = [w, y | z]$ for $z \notin V(x, y, w)$ and all $w \in X$ implies $x = y$;
- (5) $[ax, y | z] = \|x, z\|^2$ for $z \notin V(x, y)$ implies $x = ay$ for some $a > 0$, and $a = 1$ if $\|x, z\| = \|y, z\|$;
- (6) $[x, y | z] = \|x, z\| \|y, z\|$ for $z \notin V(x, y)$ implies $x = ay$ for some $a > 0$;
- (7) $[x, y | z] = \|x, z\|^2 = \|y, z\|^2 \neq 0$ for $x \neq y$ implies $z = d(x - y)$ for some $d \neq 0$;
- (6') $[x, y | z] = \|x, z\| \|y, z\|$ for $z \notin V(x, y)$ implies $\|y, z\| x = \|x, z\| y$;
- (7') $[x, y | z] = \|x, z\|^2 = \|y, z\|^2 \neq 0$ for $x \neq y$ implies $\|x, y\| \neq 0$ and $z = \pm \|x, z\| (x - y) / \|x, y\|$.

PROOF. The following implications are routine: (2) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6') $\Rightarrow$ (6) $\Rightarrow$ (3) $\Rightarrow$ (2) and (7') $\Rightarrow$ (7).

So let us prove that (3) $\Rightarrow$ (4) $\Rightarrow$ (1) $\Rightarrow$ (6'), (2) $\Rightarrow$ (1) $\Rightarrow$ (7') and (7) $\Rightarrow$ (1).

(1) $\Rightarrow$ (6'): Let $[x, y | z] = \|x, z\| \|y, z\|$ for $z \notin V(x, y)$, then $(\|x, z\| + \|y, z\|) \|y, z\| = [x + y, y | z] \leq \|x + y, z\| \|y, z\| \leq (\|x, z\| + \|y, z\|) \|y, z\|$, or $\|x + y, z\| = \|x, z\| + \|y, z\|$. Hence $\|y, z\| x = \|x, z\| y$ by (6) in Lemma 1.

(3) $\Rightarrow$ (4): Let $w = x$ in (4), then $\|x, z\|^2 = [x, y | z] \leq \|x, z\| \|y, z\|$, or $\|x, z\| \leq \|y, z\|$. If $w = y$, then $\|y, z\| \leq \|x, z\|$ similarly. Hence $\|x, z\| = \|y, z\|$ and $x = y$ by (3).

(4) $\Rightarrow$ (1): Suppose that X is not strictly convex, i.e., $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ for $z \notin V(x, y)$ and $x \neq y$, we have to show that $[w, x | z] = [w, y | z]$ for $z \notin V(x, y, w)$ and some z 's implies $x = y$. Since $\|x, z\| = \|y, z\|$ by the proof (3) $\Rightarrow$ (4) we have $[x, y | z] = \|x, z\| \|y, z\|$. As in the proof (1) $\Rightarrow$ (6') we conclude that $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$.

(2) $\Rightarrow$ (1): Let $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| = 1$ and $x \neq y$, then, with the aid of the proof (1) $\Rightarrow$ (6'), we can show easily that $[x, y | z] = \|x, z\| = \|y, z\| = 1$ implies $x = y$.

(1) $\Rightarrow$ (7'): Let $x \neq y$ and $[x, y | z] = \|x, z\|^2 = \|y, z\|^2$, so $[x, y | z] = \|x, z\| \|y, z\|$, then $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\|$ by the proof (1) $\Rightarrow$ (6'). Hence $\|x, z\| \neq 0$ and $z = \pm \|x, z\| (x - y) / \|x, y\|$ by (7) in Lemma 1.

(7) $\Rightarrow$ (1): Suppose by contrapositive that (4) in Lemma 1 does not hold, then by the proof (1) $\Rightarrow$ (6') it is easily seen that (7) does not hold, and the proof of the theorem is complete.

Motivated by the concepts of bounded linear functionals, and duality mappings on normed linear spaces [2, 9], bounded linear 2-functionals on 2-normed spaces were introduced by White [15], and associated duality mappings were defined in [3]. Let M and N be linear manifolds of X , a bounded linear 2-functional is a mapping f on $M \times N$ into real numbers such that

- (i) $f(x + x', y + y') = f(x, y) + f(x, y') + f(x', y) + f(x', y')$;
- (ii) $f(ax, by) = abf(x, y)$ for any real numbers a and b ; and
- (iii) $|f(x, y)| \leq k\|x, z\|$ for some $k \geq 0$ and all $(x, y) \in M \times N$.

In this case the norm of f is defined by

$$\|f\| = \inf\{k: |f(x, y)| \leq k\|x, y\|, (x, y) \in M \times N\}.$$

It can be shown that $|f(x, y)| \leq \|f\| \|x, y\|$ and $f(x, y) = 0$ if $x \in V(y)$ [15]. We need also a result which is similar to the Hahn-Banach theorem of functional analysis: If $x, z \in X$ and $x \notin V(z)$, then there exists a bounded linear 2-functional f on $X \times V(z)$ such that $f(x, z) = \|x, z\|$ and $\|f\| = 1$ [6, 13, 15].

The following duality mappings defined in [3] are 2-dimensional analogues of usual duality mappings on a normed linear space [2, 9]:

$$I(x, z) = \{f \in X_z^*: f(x, z) = \|f\| \|x, z\|\} \text{ and}$$

$$J(x, z) = \{f \in X_z^*: f(x, z) = \|f\| \|x, z\|, \|f\| = \|x, z\|\}$$

with duality mappings $I, J: X \times V(z) \rightarrow 2^{X_z^*}$, where X_z^* is the space of all bounded linear 2-functionals on $X \times V(z)$.

Evidently the following assertions are true: (a) $J(x, z) \subseteq I(x, z)$; (b) $I(x, z) = X_z^*$ if and only if $x \in V(z)$; (c) $I(x, z) = I(cx, dz) = cdI(x, z)$ for $c, d > 0$; (d) $0 \neq f \in I(x, z)$ for $x \notin V(z)$ implies $f \in J(cx, z)$ for some $c > 0$; and (e) If $x \notin V(z)$, then there exists an $f \in J(x, z)$ with $f \neq 0$ (by the Hahn-Banach theorem stated in above).

Let us define another type of duality mapping as follows:

DEFINITION. Let $I'(x, z)$ be the same as $I(x, z)$ which has the following additional properties:

- (i) $\|x, z\| \geq \|y, z\|$ if and only if $\|f\| \geq \|g\|$ for $z \notin V(x, y)$, $f \in I(x, z)$ and $g \in I(y, z)$; and
- (ii) $\|x, z\| \geq \|x, w\|$ if and only if $\|f\| \geq \|h\|$ for $x \notin V(z, w)$, $f \in I(x, z)$ and $h \in I(x, w)$.

It follows easily from (i) that $f \in I'(x, z) \cap I'(y, z)$ for $z \notin V(x, y)$ if and only if $f(x, z) = \|f\| \|x, z\|$, $f(y, z) = \|f\| \|y, z\|$ and $\|x, z\| = \|y, z\|$. A similar result from (ii) is obtainable.

LEMMA 2. If $0 \neq f \in I'(x, z)$, $0 \neq g \in I'(y, z)$ for $x \neq y$ and $z \notin V(x, y)$, then

- (1) $(f - g)(x - y, z) \geq 0$;
- (2) $(f - g)(x - y, z) = 0$ if and only if $f(y, z) = \|f\| \|y, z\|$, $g(x, z) = \|g\| \|x, z\|$ and $\|x, z\| = \|y, z\|$;
- (3) $(f - g)(x - y, z) = 0$ if and only if $f, g \in I'(x, z) \cap I'(y, z)$.

PROOF. (1) and (2) are straightforward computations and can be found in ([10] p. 379). Indeed, $(f - g)(x - y, z) = (\|f\| - \|g\|)(\|x, z\| - \|y, z\|) + [\|f\| \|y, z\| - f(y, z)] + [\|g\| \|x, z\| - g(x, z)] \geq 0$. (3) is consequences of (2) and a previous remark.

In a similar manner we can prove the following analogous result.

LEMMA 3. If $0 \neq f \in I'(x, z)$, $0 \neq g \in I'(x, w)$ for $z \neq w$ and $x \notin V(z, w)$, then

- (1) $(f - g)(x, z - w) \geq 0$;
- (2) $(f - g)(x, z - w) = 0$ if and only if $f(x, w) = \|f\| \|x, w\|$, $g(x, z) = \|g\| \|x, z\|$ and $\|x, z\| = \|x, w\|$;
- (3) $(f - g)(x, z - w) = 0$ if and only if $f, g \in I'(x, z) \cap I'(x, w)$.

Obviously, I' in Lemma 2 and 3 may be replaced by J . Let $\#$ denote the inclusion relation $\subseteq, \supseteq$ or

—.

THEOREM 2. If $x, y \neq 0$, then the following thirteen statements are equivalent:

- (1) X is strictly convex (in the sense of Lemma 1);

- (2) $I(x, z) \cap I(y, z) \neq \emptyset$ for $z \notin V(x, y)$ implies $x = ay$ for some $a > 0$;
- (3) $I(x, z) \# I(y, z)$ for $z \notin V(x, y)$ implies $x = ay$ for some $a > 0$;
- (4) $J(x, z) \cap J(y, z) \neq \emptyset$ for $z \notin V(x, y)$ implies $x = y$;
- (5) $J(x, z) J(y, z)$ for $z \notin V(x, y)$ implies $x = y$;
- (6) $I'(x, z) \cap I'(y, z) \neq \emptyset$ for $z \notin V(x, y)$ implies $x = y$;
- (7) $I'(x, z) I'(y, z)$ for $z \notin V(x, y)$ implies $x = y$;
- (8) If $0 \neq f \in I'(x, z)$ and $0 \neq g \in I'(y, z)$ for $x \neq y$ and $z \notin V(x, y)$, then $(f - g)(x - y, z) > 0$;
- (9) $J(x, z) \cap J(y, z) \neq \emptyset$ for $x \neq y$ implies $z = d(x - y)$ for some $d \neq 0$;
- (2') $I(x, z) \cap I(y, z) \neq \emptyset$ for $z \notin V(x, y)$ implies $\|y, z\|x = \|x, z\|y$;
- (3') $I(x, z) \# I(y, z)$ for $z \notin V(x, y)$ implies $\|y, z\|x = \|x, z\|y$;
- (8') If $0 \neq f \in J(x, z)$ and $0 \neq g \in J(y, z)$ for $x \neq y$ and $z \notin V(x, y)$, then $(f - g)(x - y, z) > 0$;
- (9') $J(x, z) \cap J(y, z) \neq \emptyset$ for $x \neq y$ implies $\|x, y\| \neq 0$ and $z = \pm \|x, z\| (x - y) / \|x, y\|$.

PROOF. The proof of $(2') \Rightarrow (2) \Rightarrow (3)$, $(2') \Rightarrow (3') \Rightarrow (3)$ and $(9') \Rightarrow (9)$ are trivial. Equivalences of (1), (4), (5), (6) and (7) are clear after we verify the implications $(3) \Rightarrow (1) \Rightarrow (2')$. (8') is, of course, a special case of (8).

$(1) \Rightarrow (2')$: Let $0 \neq f \in I(x, z) \cap I(y, z) = I(x, z) \cap I(\|x, z\|y / \|y, z\|, z)$, then $\|f\| \|x + (\|x, z\|y / \|y, z\|), z\| \geq f(x + (\|x, z\|y / \|y, z\|), z) = 2\|f\| \|x, z\| \geq \|f\| \|x + (\|x, z\|y / \|y, z\|), z\|$, or $\|x + (\|x, z\|y / \|y, z\|), z\| = 2\|x, z\|$ and hence $\|y, z\|x = \|x, z\|y$ by (5) in Lemma 1.

$(3) \Rightarrow (1)$: Without loss of generality we may assume that $0 \neq f \in I(x, z) \subseteq I(y, z)$ in (3). Suppose that $\|x + y, z\| = \|x, z\| + \|y, z\|$ and $x \neq by$ for all $b > 0$, i.e., the negation of (3) in Lemma 1, we have to show that $f \in I(x, z) \subseteq I(y, z)$ implies $x \neq by$ for all $b > 0$. This follows from the relation $\|f\| \|x + y, z\| \geq f(x + y, z) = \|f\| (\|x, z\| + \|y, z\|) \geq \|f\| \|x + y, z\|$, or $\|x + y, z\| = \|x, z\| + \|y, z\|$.

$(6) \Rightarrow (8)$: Let $0 \neq f \in I'(x, z)$, $0 \neq g \in I'(y, z)$, $x \neq y$, $z \notin V(x, y)$ and $(f - g)(x - y, z) = 0$, then $f \in I'(x, z) \cap I'(y, z)$ by Lemma 2, and $x \neq y$. Thus (6) does not hold.

$(8) \Rightarrow (6)$: If $f \in I'(x, z) \cap I'(y, z)$ and if $x \neq y$, then $0 = (f - f)(x - y, z) > 0$ by (8) yielding a contradiction.

$(1) \Rightarrow (9')$: For $x \neq y$ let $0 \neq f \in J(x, z) \cap J(y, z)$, then $\|x, z\| = \|y, z\| = \|f\| \neq 0$. It follows easily that $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| \neq 0$. Hence $\|x, y\| \neq 0$ and $z = \pm \|x, z\| (x - y) / \|x, y\|$ by (7) in Lemma 1.

$(9) \Rightarrow (1)$: Consider the negation of (4) in Lemma 1, i.e., $\frac{1}{2}\|x + y, z\| = \|x, z\| = \|y, z\| \neq 0$, $x \neq y$ and $z \neq d(x - y)$ for all $d \neq 0$, then as in the proof $(1) \Rightarrow (9')$ we can easily conclude that (9) does not hold.

REMARKS. (a) That $J(x, z) \cap J(y, z) \neq \emptyset$ in (9) and (9') above may be replaced, of course, by $J(x, z) \# J(y, z)$ without any other change in the statements; (b) J in (9) and (9') may be replaced by I' if $\|x, z\|$ or $\|y, z\| \neq 0$ in addition to the conditions; (c) Though (2) appeared in ([3] Theorem 1), our proof is direct and much simpler. (4) is in ([3] Corollary 3). (8) was discussed in ([10] Theorem 2.5) with a different type of duality mapping; (d) Note that a duality mapping which satisfies the statement (8) is said to be strictly monotone [10] (cf. [2, 9]). In other words, X is strictly convex if and only if I' or J is strictly monotone.

3. STRICTLY 2-CONVEX 2-NORMED SPACES.

According to [6] X is said to be strictly 2-convex if $\|x+z, y+z\|/3 = \|x, y\| = \|y, z\| = \|z, x\| = 1$ implies $z = x + y$. We now turn to the investigation of this space in terms of 2-semi-inner products and duality mappings. To this end we require first the next result which is a portion of Theorem 2 in [11].

LEMMA 4. The following four statements are equivalent:

- (1) X is strictly 2-convex;
- (2) $\|x+z, y+z\| = \|x, y\| + \|y, z\| + \|z, x\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$ for some $b, c > 0$;
- (3) $\|bx+z, cy+z\| = 3\|bx, z\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$, where $b = \|y, z\|/\|x, y\|$ and $c = \|x, z\|/\|x, y\|$.
- (4) $\|x+z, y+z\| = \|x, y\| + \|y, z\| + \|z, x\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$, where b and c are as in (3).

In order to be able to prove the next theorem we shall use one of the basic properties of a 2-norm that $\|ax + by, y\| = |a| \|x, y\|$ for any real numbers a and b [7].

THEOREM 3. The following five statements are equivalent:

- (1) X is strictly 2-convex (in the sense of Lemma 4);
- (2) $[-x, y \mid y+z] = (\|x, y\| + \|x, z\|) \|y, z\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$ for some $b, c > 0$;
- (3) $\frac{1}{2}[-x, y \mid y+z] = \|x, y\|^2 = \|y, z\|^2 = \|z, x\|^2 \neq 0$ implies $z = x + y$;
- (4) $\frac{1}{2}[-x, y \mid y+z] = \|x, y\| = \|y, z\| = \|z, x\| = 1$ implies $z = x + y$;
- (2') $[-x, y \mid y+z] = (\|x, y\| + \|x, z\|) \|y, z\|$ for $\|x, y\| \|y, z\| \|z, x\| \neq 0$ implies $z = bx + cy$, where $b = \|y, z\|/\|x, y\|$ and $c = \|x, z\|/\|x, y\|$.

PROOF. The following implications are trivial: $(2') \Rightarrow (2) \Rightarrow (3) \Rightarrow (4)$.

$(1) \Rightarrow (2')$: If $[-x, y \mid y+z] = (\|x, y\| + \|x, z\|) \|y, z\|$, then $(\|x, y\| + \|y, z\| + \|z, x\|) \|y, z\| = [y-x, y \mid y+z] \leq \|y-x, y+z\| \|y, z\| = \|(y+z) - (x+z), y+z\| \|y, z\| = \|x+z, y+z\| \|y, z\| \leq (\|x, y\| + \|y, z\| + \|z, x\|) \|y, z\|$, or $\|x+z, y+z\| = \|x, y\| + \|x, z\| + \|y, z\|$ and the result follows by (4) in Lemma 4.

$(4) \Rightarrow (1)$: If $\|x+z, y+z\|/3 = \|x, y\| = \|y, z\| = \|z, x\| = 1$ and $z \neq x + y$, we have to show that $\frac{1}{2}[-x, y \mid y+z] = \|x, y\| = \|y, z\| = \|z, x\| = 1$ implies $z = x + y$. But this is clear from the proof in above.

THEOREM 4. In the following let $I(u, v)$, $J(u, v)$ and $I'(u, v)$ be defined as in the previous section, and let $u \notin V(v)$, then the following seven statements are equivalent:

- (1) X is strictly 2-convex (in the sense of Lemma 4);
- (2) $I(x, y) \cap I(x, z) \cap I(z, y) \neq \emptyset$ implies $z = bx + cy$ for some $b, c > 0$;
- (3) $J(x, y) \cap J(x, z) \cap J(z, y) \neq \emptyset$ implies $z = x + y$;
- (4) $I'(x, y) \cap I'(x, z) \cap I'(z, y) \neq \emptyset$ implies $z = x + y$;
- (5) If $0 \neq f \in I'(x, y)$, $0 \neq g \in I'(x, z)$ and $0 \neq h \in I'(z, y)$ for $z \neq x + y$, then $(f-h)(x-z, y)$ and $(f-g)(x, y-z) > 0$;
- (2') $I(x, y) \cap I(x, z) \cap I(z, y) \neq \emptyset$ implies $z = bx + cy$ for $b = \|y, z\|/\|x, y\|$ and $c = \|x, z\|/\|x, y\|$;

(5') If $0 \neq f \in J(x, y)$, $0 \neq g \in J(x, z)$ and $0 \neq h \in J(z, y)$ for $z \neq x + y$, then $(f - h)(x - z, y)$ and $(f - g)(x, y - z) > 0$.

PROOF. That (2') $\Rightarrow$ (2) is trivial. (5') is a special case of (5), and it is clear that we need to verify that (2) $\Rightarrow$ (1) $\Rightarrow$ (2') and (4) $\Leftrightarrow$ (5) only.

(1) $\Rightarrow$ (2'): Let $0 \neq f \in I(x, y) \cap I(x, z) \cap I(z, y) = I(bx, cy) \cap I(bx, z) \cap I(z, cy)$, where $b = \|y, z\| / \|x, y\|$ and $c = \|x, z\| / \|x, y\|$, then $\|f\| \|bx + z, cy + z\| \leq \|f\| (\|bx, cy\| + \|bx, z\| + \|z, cy\|) = f(bx + z, cy + z)$, $\leq \|f\| \|bx + z, cy + z\|$, or $\|bx + z, cy + z\| = \|bx, cy\| + \|bx, z\| + \|z, cy\| = 3\|bx, z\|$, and $\|x, y\| \|y, z\| \|z, x\| \neq 0$ by assumption. So $z = bx + cy$ by (3) in Lemma 4.

(2) $\Rightarrow$ (1): Consider the negation of (2) in Lemma 4, i.e., $\|x + z, y + z\| = \|x, y\| + \|x, z\| + \|z, y\|$, $\|x, y\| \|y, z\| \|z, x\| \neq 0$ and $z \neq bx + cy$ for all $b, c > 0$, we have to show that $0 \neq f \in I(x, y) \cap I(x, z) \cap I(z, y)$ implies $z \neq bx + cy$ for all $b, c > 0$. This follows from the relation $\|f\| \|x + z, y + z\| \geq f(x + z, y + z) = \|f\| (\|x, y\| + \|x, z\| + \|z, y\|) \geq \|f\| \|x + z, y + z\|$, or $\|x + z, y + z\| = \|x, y\| + \|x, z\| + \|z, y\|$.

(4) $\Rightarrow$ (5): Let $0 \neq f \in I'(x, y)$, $0 \neq g \in I'(x, z)$, $0 \neq h \in I'(z, y)$, $(f - h)(x - z, y) = 0 = (f - g)(x, y - z)$ and $z \neq x + y$, i.e., the negation of (5), then $f \in I'(x, y) \cap I'(z, y) \cap I'(x, z)$ by Lemma 2 and 3, and $z \neq x + y$. Thus (4) does not hold.

(5) $\Rightarrow$ (4): If $f \in I'(x, y) \cap I'(x, z) \cap I'(z, y)$ and suppose that $z \neq x + y$, then $0 = (f - f)(x - z, y) > 0$ by (5) yielding a contradiction, and the proof of the theorem is complete.

REMARK. (2) in Theorem 4 appeared in ([4] Theorem 1.2) except that the domain of the duality mapping I has been changed. The change is unnecessary.

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