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R-CONTINUOUS FUNCTIONS

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ABSTRACT. A strong form of continuity of functions between topological spaces is introduced and studied. It is shown that in many known results, especially closed graph theorems, functions under consideration are R-continuous. Several results in the literature concerning strong continuity properties are generalized and/or improved.

KEY WORDS AND PHRASES. R-continuity, continuity, closed graph property, rim compactness.
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1. INTRODUCTION.

The purpose of this paper is to introduce a strong form of continuity of functions between topological spaces and study its properties. We call this new class of functions, R-continuous functions, because it is closely related to the class of regular topological spaces and is instrumental in discussing preservation of regularity under mappings. Since R-continuous functions into T_1 spaces have closed graphs it is not surprising that in many familiar closed graph theorems considered, functions are R-continuous. For instance, a well known result that closed graph functions into compact spaces are continuous is strengthened by showing that they are actually R-continuous. It is also observed that R-continuous functions are encountered in various standard settings. Among others, we show that continuous functions from locally compact Hausdorff spaces into Hausdorff spaces are R-continuous. We also establish numerous basic results about R-continuity as well as obtain several improvements of known results.

2. PRELIMINARIES.

By $f: X \to Y$ is meant an arbitrary function between topological spaces. For a subset A of a space $X, c\ell A$, int A, and BdA denote the closure, interior and boundary of A in X, respectively. Recall that A is regular open if $A =$ int $c\ell A$. The family of regular open sets in X is a base for a topology on X, denoted by τ_s. For $A \subset X$, the θ-closure (Veličko [1]) of A in X is $c\ell_\theta A = \{x \in X: c\ell U \cap A \neq \phi$, for each open $U \subset X$ with $x \in U\}$. A is θ-closed if $A = c\ell_\theta A$ and θ-open sets in X are complements of θ-closed sets. For a space (X, τ), the space (X, τ_θ) is the set X with the topology τ_θ of θ-open sets. It is clear that $\tau_\theta \subset \tau_3 \subset \tau$ and that a space is regular if and only if $\tau = \tau_\theta$. Recall now that a function $f: X \to Y$ is strongly θ-continuous (Noiri [2]) (resp. θ-continuous (Fomin [3]), weakly continuous (Levine [4])) if for each $x \in X$ and each open set V containing f(x), there exists an open set U containing x such that $f(c\ell U) \subset V$ (resp. $f(c\ell U) \subset c\ell V$,

$f(U) \subset c\ell V)$. Note that $f:(X,\tau) \to Y$ is strongly θ-continuous if and only if $f:(X,\tau_\theta) \to Y$ is continuous. It is known that strong θ-continuity ====> continuity ====> θ-continuity ====> weak continuity and that none of these implications is reversible. However, if the domain of a continuous function is regular then the function is strongly θ-continuous and if the range of a weakly continuous function is regular then the function is strongly θ-continuous. If $\mathfrak{F}$ is a filter base on a space X we denote by ad $\mathfrak{F}$ the adherence of $\mathfrak{F}$, that is $\cap\{c\ell F: F \in \mathfrak{F}\}$ and by $c\ell\mathfrak{F}$ the filter base $\{c\ell F: F \in \mathfrak{F}\}$. $\mathbf{R}$ stands for the set of real numbers.

3. BASIC PROPERTIES OF R-CONTINUITY.

DEFINITION 3.1. A function $f: X \to Y$ is R-continuous at $x \in X$ if for each open set V containing f(x), there exists an open set U containing x such that $c\ell f(U) \subset V$.

A function $f: X \to Y$ is R-continuous if it is R-continuous at each $x \in X$. Note that a function $f: X \to Y$ is R-continuous if open sets U and V in Definition 3.1 are replaced by a basic open set and a subbasic open set, respectively. Also, it is clear that R-continuous functions are continuous. Moreover, R-continuity implies strong θ-continuity, while the converse is not true in general. We will first characterize R-continuity and then provide some examples to distinguish between continuity, strong θ-continuity and R-continuity.

THEOREM 3.1. Let $f: X \to Y$ be a function. Then the following are equivalent.

(a) f is R-continuous.

(b) For every filter base $\mathfrak{F}$ on X, if $\mathfrak{F}$ converges to $x \in X$ then $c\ell f(\mathfrak{F})$ converges to $f(x)$.

(c) For every $x \in X$ and for every open set V containing $f(x)$ there exists an open set of U containing x such that $c\ell f(c\ell U) \subset V$.

(d) For every $x \in X$ and for every closed set F in Y with $f(x) \in F$, there exist open sets U and V such that $x \varepsilon U, F \subset V$ and $f(c\ell U) \cap V = \phi$.

(e) For every $x \in X$ and for every closed set F in Y with $f(x) \in F$, there exist open sets U and V such that $x \in U, F \subset V$ and $f(U) \cap V = \phi$.

PROOF. We prove only (a) ===> (b), (b) ===> (c) and (e) ==> (a) since (c) ===> (d) and (d) ===> (e) are obvious.

(a) ===> (b). Let $x \in X$, let $\mathfrak{F}$ be a filter base on X converging to x and let V be an open set in Y containing f(x). Since f is R-continuous, there exists an open set U containing x such that $c\ell f(U) \subset V$. Since $\mathfrak{F}$ converges to x there is an $F \in \mathfrak{F}$ such that $F \subset U$. Hence $c\ell f(F) \subset c\ell f(U) \subset V$ and so, the filter $c\ell f(\mathfrak{F})$ converges $f(x)$.

(b) ===> (c). Let $x \in X$ and let V be an open set in Y containing f(x). The family $\mathfrak{F} = \{U_x: U_x$ is an open in X containing $x\}$ is a filter base converging to x. By hypothesis $c\ell f(\mathfrak{F}) = \{c\ell f(U_x): U_x$ is an open set in X containing $x\}$ converges to f(x). So, there exists an open set U_x containing x such that $c\ell f(U_x) \subset V$. This implies that f is R-continuous and consequently $f(c\ell U_x) \subset c\ell f(U_x)$. So, $c\ell f(c\ell U_x) \subset V$.

(e) ===> (a). Let $x \in X$ and let V be an open set in Y containing f(x). Then $Y - V$ is closed in Y and $f(x) \notin Y - V$. Put F = Y - V. By hypothesis, there exist open sets U and W such that $x \in U, F \subset W$ and $f(U) \cap W = \phi$ and so, $f(U) \subset Y - W$. Therefore, $c\ell f(U) \subset c\ell f(Y - W) = Y - W \subset V$ and hence f is R-continuous.

The following example given in MacDonald and Willard [5] shows that quotient mappings from regular spaces onto Hausdorff spaces are not necessarily R-continuous. Note that the function in this example is strongly θ-continuous and has a closed graph.

EXAMPLE 3.1. Let $X = X_1 \cup X_2$, where

$$X_1 = \{(0,0)\} \cup \{(0,\tfrac{1}{n}+\tfrac{1}{n^m}) : n = 1,2,...;\ m = 2,3,...\}$$

and

$$X_2 = \{(1,\tfrac{1}{n}) : n = 1,2,...\} \cup \{(1,\tfrac{1}{n}+\tfrac{1}{n^m}) :\ n = 1,2,...;\ m = 2,3,...\}$$

and X is considered as a subspace of the plane. By identifying the points $(0,\frac{1}{n}+\frac{1}{n^m})$ and $(1,\frac{1}{n}+\frac{1}{n^m})$ for $n = 1,2,...$, and $m = 2,3,...$, we obtain an open quotient map onto a non-regular Hausdorff space. The range of the surjection is not regular, since the closed set $\{(1,\frac{1}{n})$: $n = 1,2,...\}$ and the point (0,0) cannot be separated by open sets. As a consequence of our Theorem 6.1 we have that the range of an R-continuous open surjection is regular. Therefore, the quotient map is not R-continuous, while being strongly θ-continuous.

It is clear that a continuous function into a regular space is R-continuous. This observation slightly generalizes Theorem 8 of Long and Herrington [6]. Furthermore, the following characterization of regular spaces is readily established.

PROPOSITION 3.1. Let Y be a space. Then the following are equivalent.

(a) Y is regular.

(b) The identity function $i: Y \to Y$ is R-continuous (resp. strongly θ-continuous).

(c) For every space X every continuous function from X into Y is R-continuous (resp. strongly θ-continuous).

To offer another sufficient condition for R-continuity we need the concept of subweak closedness of functions. A function $f: X \to Y$ is said to be subweakly closed (Jankovič and Rose [7]) if there exists a base $\mathfrak{B}$ for the topology on X such that $c\ell f(U) \subset f(c\ell U)$ for every $U \in \mathfrak{B}$. The importance of this notion lies in the fact that the projections from the product of spaces onto the co-ordinate spaces are subweakly closed.

PROPOSITION 3.2. If $f: X \to Y$ is strongly θ-continuous and subweakly closed, then f is R-continuous.

PROOF. Since f is subweakly closed, there exists a base $\mathfrak{B}$ for the topology on X such that $c\ell f(U) \subset f(c\ell U)$, for every $U \in \mathfrak{B}$. The strong θ-continuity of f gives that for every $x \in X$ and every open V containing f(x) there exists $U \in \mathfrak{B}$ such that $f(c\ell U) \subset V$. The result follows by combining the previous two facts.

To see that the converse of the Proposition 3.2. does not hold, consider the identity function $i: (\mathbf{R}, \tau) \to (\mathbf{R}, \sigma)$, where τ is the usual topology on $\mathbf{R}$ and σ is the indiscrete topology of $\mathbf{R}$.

THEOREM 3.2. If $f: X \to Y$ is continuous, X is locally compact Hausdorff, and Y is Hausdorff, then f is R-continuous.

PROOF. Since X is regular and f is continuous, f is strongly θ-continuous. The local compactness of X implies that there is a base $\mathfrak{B}$ for the topology on X consisting of open sets with compact closures. Let $U \in \mathfrak{B}$. Since f is continuous, $f(c\ell U)$ is compact in Y and hence closed because Y is Hausdorff. Therefore, f is subweakly closed and so, by Proposition 3.2, f is R-continuous.

Notice that the space X in Example 3.1 is not locally compact.

In the light of Proposition 3.1 it is reasonable to expect that our strong form of continuity will produce a weak form of regularity on the range of a function. Recall that a space X is R_0 (Davis

[8]), if for every open set U in X and every $x \in U$, $c\ell\{x\} \subset U$. It is known that regular spaces are R_0 while the converse is not true and that a space is T_1 if and only if it is R_0 and T_0.

PROPOSITION 3.3. If $f: X \to Y$ is an R-continuous surjection, then Y is R_0.

PROOF. Let $y \in Y$ and let V be an open set containing y. There exists an $x \in X$ such that $y = f(x)$. Since f is R-continuous, there is an open set U containing x such that $c\ell f(U) \subset V$. This implies $c\ell\{y\} \subset V$ and hence Y is R_0.

The previous result enables us to construct in a simple way examples of strongly θ-continuous and hence continuous functions which are not R-continuous. The following example is in that sense instructive.

EXAMPLE 3.2. Let τ be the discrete topology on $\mathbf{R}$ and σ the left ray topology on $\mathbf{R}$. The identity function $i: (\mathbf{R}, \tau) \to (\mathbf{R}, \sigma)$ is strongly θ-continuous but not $\mathbf{R}$-continuous, since (R, σ) is not R_0.

Next we state several basic properties of R-continuous functions without proofs since the proofs are similar to those in the case of continuous functions.

PROPOSITION 3.4.

(a) If $f: X \to Y$ is continuous (resp. R-continuous) and $g: Y \to Z$ is R-continuous, then the composition $gof: X \to Z$ is R-continuous.

(b) If $f: X \to Y$ is R-continuous and $g: Y \to Z$ is continuous and closed, then the composition $gof: X \to Z$ is R-continuous.

(c) Let $f: X \to Y$ and $g: Y \to Z$ be functions. If the composition $gof: X \to Z$ is R-continuous and f is an open surjection, then g is R-continuous.

(d) If $f: X \to Y$ is an R-continuous function and $A \subset X$, then the restriction $f/A: A \to B$ is R-continuous, for any B with $f(A) \subset B \subset Y$.

(e) If $f: X \to Y$ is a function such that $f_i = f/U_i: U_i \to Y$ is R-continuous for every $i \varepsilon I$, where $\{U_i: i \varepsilon I\}$ is an open cover of X, then f is R-continuous.

(f) If each $X_i \neq \phi$, then the product function $\Pi f_i: \Pi X_i \to \Pi Y_i$ is R-continuous iff each $f_i: X_i \to Y_i$ is R-continuous.

4. R-CONTINUITY AND CLOSED GRAPH PROPERTY.

Since the class of R-continuous functions into T_1 spaces is properly contained in the class of functions with closed graphs, a natural question arises. Under what conditions are closed graph functions R-continuous? In answering this question we will strengthen some known closed graph theorems. But first we will show that a function possessing a strong continuity property will also possess some strong form of closed graph property if its range satisfies a certain, usually weak, separation axiom.

Recall that a function $f: X \to Y$ has a θ-closed graph (resp. θ-closed graph with respect to X) (Hamlett and Herrington [9]) if for every (x,y) $\notin$ G(f) (G(f) denotes the graph of f), there exist open sets U and V containing x and y, respectively, such that $(c\ell U \times c\ell V) \cap G(f) = \phi$ (resp. $(c\ell U \times V) \cap G(f) = \phi$), or equivalently, $f(c\ell U) \cap c\ell V = \phi$ (resp. $f(c\ell U) \cap V = \phi$). Note that $f: X \to Y$ has a θ-closed graph if and only if $G(f) = c\ell_\theta G(f)$.

Our first result is a variation of the well known and useful fact that continuous functions into Hausdorff spaces have closed graphs (actually, θ-closed graphs with respect to the domains).

THEOREM 4.1. If $f: X \to Y$ is an R-continuous (resp. strongly θ-continuous) function and Y is T_1 (resp. Hausdorff), then f has a θ-closed graph with respect to X (resp. θ-closed graph).

PROOF. The first part of the result follows from the facts that R-continuous functions into T_1 spaces have closed graphs and that continuous functions with closed graphs have θ-closed graphs with respect to the domains. To show the second part, let $x \in X$, $y \in Y$ and $y \neq f(x)$. Since Y is Hausdorff, there exist open sets V and W such that $f(x) \in V$, $y \in W$ and $V \cap c\ell W = \phi$. The strong θ-continuity of f implies that there exists an open set U containing x such that $f(c\ell U) \subset V$. Therefore, $f(c\ell U) \cap c\ell W = \phi$ and f has a θ-closed graph.

The parenthetical part of Theorem 4.1 improves Theorem 12 of (Long and Herrington [6]) where it was shown that a strongly θ-continuous function into a Hausdorff space has a θ-closed graph with respect to the domain. We remark that another improvement of this result can be obtained by replacing the hypothesis of strong θ-continuity with θ-continuity.

Note that the function considered in Example 3.1 has a θ-closed graph but is not R-continuous. Therefore, the converse of the first part of Theorem 4.1 does not hold. Our next example shows that there is a function with a θ-closed graph into a Hausdorff space which is not strongly θ-continuous.

EXAMPLE 4.1. Let τ be the usual topology of $\mathbf{R}$ and σ be the topology $\{U - A : U \varepsilon \tau$ and $A \subset \{\frac{1}{n} : n = 1, 2, ...\}\}$ on $\mathbf{R}$. Clearly, $\tau \subset \sigma$ and $(\mathbf{R}, \sigma)$ is an Urysohn space. The identity function $i : (\mathbf{R}, \tau) \to (\mathbf{R}, \sigma)$ has a θ-closed graph, but is not strongly θ-continuous since $\sigma_\theta = \sigma_3 = \tau$ and i is not continuous.

By Theorem 4.1 and Proposition 3.3 it easily follows that R-continuous functions onto T_0 spaces have θ-closed graphs with respect to the domains.

The proof of the next proposition is straight forward and hence is omitted.

PROPOSITION 4.1. If $f : X \to Y$ is an injection with a θ-closed graph with respect to X, then X is Hausdorff.

Combining Proposition 4.1 and the first part of Theorem 4.1 we may get a sufficient condition for the domain of a function to be Hausdorff. But, it is not difficult to see that a strongly θ-continuous injection into a T_0 space has a Hausdorff domain. This observation improves Theorem 4 of [6] and also implies the known fact that T_0 regular spaces are T_3.

We now turn our attention to the closed graph theorems. The following closed graph theorem was established independently by Rose [10] and Noiri [11]: every closed graph weakly continuous function into a rim-compact space is continuous. This result was generalized in (Janković [12]). We will show that a closed graph weakly continuous function into a rim-compact space has a strong continuity property, namely, it is R-continuous. Actually our result states that closed graph continuous functions into rim-compact spaces are R-continuous. As an immediate consequence we will obtain a strengthening of the well-known theorem that functions with closed graphs into compact spaces are continuous. Recall that a space is called rim-compact, if its topology has an open basis of sets whose boundaries are compact.

THEOREM 4.2. If $f : X \to Y$ is a weakly continuous function with a closed graph and Y is rim-compact, then f is R-continuous.

PROOF. In order to shorten the proof we assume that f is continuous. Let $\mathfrak{F}$ be a filter base on X converging to x and let V be a basic open set such that $f(x) \in V$ and $\mathbf{B}d(V)$ compact. Assume that $Bd(V) \neq \phi$. Otherwise the proof is complete. Since $f(\mathfrak{F})$ converges to $f(x)$, there exists a $F^* \in \mathfrak{F}$ such that $f(F^*) \subset V$. Let $\mathfrak{F}* = \{F^* \cap F : F \in \mathfrak{F}\}$. Clearly, $\mathfrak{F}$ converges to x, $f(\mathfrak{F}^*)$ converges to $f(x)$ and $c\ell f(F) \subset c\ell V$ for every $F \in \mathfrak{F}*$. In order to establish that $c\ell f(\mathfrak{F})$ converges to $f(x)$ it is enough to show that $c\ell f(\mathfrak{F}^*)$ converges to $f(x)$. Suppose that this is not the case.

Then $c\ell f(F) \cap (Y - V) \neq \phi$ for every $F \in \mathfrak{F}^*$, since $c\ell f(F) \subset c\ell V$, $c\ell f(F) \cap \boldsymbol{Bd}(V) \neq \phi$. The compactness of $\boldsymbol{Bd}(V)$ implies there is a $y \varepsilon a d c\ell f(\mathfrak{F}^*) \cap \boldsymbol{Bd}(V)$. This gives $y \varepsilon a d f(\mathfrak{F}^*) \cap \boldsymbol{Bd}(V)$. Since $\mathfrak{F}^*$ converges to x and f has a closed graph, $y = f(x)$. So, $f(x) \varepsilon \boldsymbol{Bd}(V)$. This contradiction establishes that $c\ell f(\mathfrak{F})$ converges to $f(x)$. By Theorem 2.2, f is R-continuous.

In [7] it was observed that closed graph functions into compact spaces are strongly θ-continuous. By Theorem 4.2 we have an improvement of this result.

COROLLARY 4.1. If $f: X \to Y$ is a function with a closed graph and Y is compact, then f is R-continuous.

In [6], Long and Herrington established that a θ-continuous function $f: X \to Y$ with a θ-closed graph with respect to X into the rim-compact space Y is strongly θ-continuous. [Theorem 10] and also, that a function $f: X \to Y$ with a θ-closed graph with respect to X into compact space Y is strongly θ-continuous [Theorem 11]. Our Theorem 4.2 and Corollary 4.1 considerably improve their results.

Combining Theorem 4.1. and Corollary 4.1, we obtain the following characterization of R-continuous functions into T_1 compact spaces.

COROLLARY 4.2. Let $f: X \to Y$ be a function and let Y be a T_1 compact space. Then f is R-continuous if and only if f has a closed graph.

Theorem 9 of [6] and our Theorem 4.1 give a similar characterization of strong θ-continuity.

COROLLARY 4.3. Let $f: X \to Y$ be a function and let Y be a minimal Hausdorff space. Then f is strongly θ-continuous if and only if f has a θ-closed graph.

We close this section by pointing out that a closed graph function from a first countable space into a countably compact space is R-continuous. This result strengthens Theorem I.1.11 in [9].

5. FURTHER PROPERTIES OF R-CONTINUOUS AND STRONGLY θ-CONTINUOUS FUNCTIONS.

It is a well known fact that if continuous functions f and g which may, into a Hausdorff space, agree on a dense subset of the domain, then they coincide. In this section we obtain similar results for R-continuous and strongly θ-continuous functions.

THEOREM 5.1. If $f: X \to Z$ is a function with a θ-closed graph with respect to X and $g: Y \to Z$ is a strongly θ-continuous function, then the set $\{(x,y) \in X \times Y: f(x) = g(y)\}$ is θ-closed.

PROOF. Let $A = \{(x,y) \in X \times Y: f(x) = g(y)\}$ and let $(x,y)A$. Then $f(x) \neq g(y)$ and $(x, g(y)) \in (X \times Z) - G(f)$. Since f has a θ-closed graph with respect to X, there exists open sets U and W containing x and $g(y)$, respectively, such that $f(c\ell U) \cap W = \phi$. The strong θ-continuity of g implies that there is an open set V containing y such that $g(c\ell V) \subset W$. Therefore $f(c\ell U) \cap g(c\ell V) = \phi$. We claim that $(c\ell U \times c\ell V) \cap A = \phi$ and hence $A = c\ell_\theta A$. Suppose that there is $a(x,y) \in (c\ell U \times c\ell V) \cap A$. A. Then $f(x) = g(y), f(x) \in f(c\ell U)$ and $g(y) \in g(c\ell V)$ implying $f(c\ell U) \cap g(c\ell V) \neq \phi$. This contradiction establishes the proof.

By analyzing the proof of Theorem 5.1, it is easy to see that there are numerous variations of Theorem 4.1. The most important one is certainly the following: if $f: X \to Z$ has a closed graph and $g: Y \to Z$ is continuous, then the set $\{(x,y) \in X \times Y: f(x) = g(y)\}$ is closed. Other possibilities are left to the reader.

We now cite some results that follow from the previously established results or may be obtained in similar manner.

COROLLARY 5.1. If $f: X \to Z$ is R-continuous (resp. θ-continuous) $g: Y \to Z$ is strongly θ-continuous, and Z is T_1 (resp. Hausdorff) then the set $\{(x,y) \in X \times Y: f(x) = g(y)\}$ is θ-closed.

THEOREM 5.2. If $f: X \to Y$ has a θ-closed graph with respect to X (resp. θ-closed graph) and $g: X \to Y$ is strongly θ-continuous (resp. θ-continuous) then the set $\{x \varepsilon X: f(x) = g(x)\}$ is θ-closed.

The second part of the next corollary improves Theorem 2 of [6].

COROLLARY 5.2. If $f: X \to Y$ is R-continuous (strongly θ-continuous) and $g: X \to Y$ is strongly θ-continuous (θ-continuous) and Y is T_1 (Hausdorff), then $\{x \in X: f(x) = g(x)\}$ is θ-closed.

COROLLARY 5.3. If $f, g: X \to Y$ are R-continuous (resp. strongly θ-continuous) functions which agree on a θ-dense subset A of X (i.e. $c\ell_\theta A = X$) and Y is T_1 (resp. Hausdorff), then $f = g$.

6. PRESERVATION OF REGULARITY.

As we have mentioned in the introduction one of the reasons for introducing the concept of R-continuity is its intrinsic relationship with the regularity separation axiom. As observed in Willard [13], very strong conditions must be imposed on a surjection so that it preserves regularity. Chaber [14] established that T_3 is preserved by continuous closed open surjections. We obtain this result by use of R-continuity. First, we need to recall that a function $f: X \to Y$ is almost-open [15] if $f(U) \subset \text{ int } c\ell f(U)$ for every (basic) open set U in X. Open functions are almost open while the converse is not true even in the presence of R-continuity as the following example shows.

EXAMPLE 6.1. Let τ be the usual topology on $\mathbf{R}$ and σ be the simple extension of τ by the set of rationals $\mathbf{Q}$ i.e. $\sigma = \{(U \cap \mathbf{Q}) \cup V: U, V \varepsilon \tau\}$. The identity function $i: (X, \sigma) \to (X, \tau)$ is R-continuous and almost open but not open.

THEOREM 6.1. If $f: X \to Y$ is an R-continuous almost-open surjection, then Y is regular.

PROOF. Let $y \in Y$ and let V be an open set containing y. There exists an $x \in X$ such that $y = f(x)$ and since f is R-continuous there is an open set U containing x such that $c\ell f(U) \subset V$. The almost openness of f gives $f(U) \subset \text{ int } c\ell f(U)$ and consequently, $y \varepsilon f(U) \subset W \subset c\ell W \subset V$, where $W = \text{ int } c\ell f(U)$. Hence, Y is regular.

COROLLARY 6.1. Regularity is preserved under continuous open subweakly closed surjections.

PROOF. Since X is regular and $f: X \to Y$ is continuous, f is strongly θ-continuous. By Proposition 3.2 it follows that f is R-continuous and by Theorem 6.1 that Y is regular.

REMARK 6.1. We point out that replacing the hypothesis open in Corollary 6.1 with almost-open does not strengthen the result, since almost open subweakly closed functions from regular spaces are open. Moreover, if $f: X \to Y$ is a subweakly closed function from a regular space X such that $f(U) \subset \text{ int } c\ell f(c\ell U)$ for every open U in X, then f is open. Let us agree to call a function $f: X \to Y$ almost weakly open if $f(U) \subset \text{ int } c\ell f(c\ell U)$ for every (basic) open set U in X. Clearly, almost-open ===> almost-weakly-open and almost-weakly-open functions from regular spaces are almost open. Also, the class of almost-open functions is properly contained in the class of almost-weakly-open functions. To see this recall first that a function $f: X \to Y$ is weakly-open (Rose [16]) if $f(U) \subset \text{ int } f(c\ell U)$ for every (basic) open set U in X. It was shown in [16] that weak openness and almost openness are independent. Since weak openness obviously implies almost weak openness, we conclude that almost weak openness is a common generalization of both weak openness and almost openness. Now, it is clear that replacing "open" in Corollary 6.1 with "weak open" also does not strengthen the result. The proofs of previous observations are left to the reader.

COROLLARY 6.2. If $f: X \to Y$ is a continuous open closed surjection and X is T_3, then Y is T_3.

PROOF. Note that T_1 is preserved by closed surjections and apply Corollary 6.1.

We remark that Corollary 6.2 improves Theorem 14.6 in [13]. Also, it may be noted that the well-known result that each factor of a non-empty regular (T_1) product space is regular (T_1) follows from Corollary 6.1 (since the projections are continuous, open, subweakly closed surjections). That the product of regular (T_1) spaces is regular (T_1) follows from Proposition 3.4 (f) by setting $X_i = Y_i$, letting f_i be the identity function on X_i, and then using the equivalence of (a) and (b) from Proposition 3.1.

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